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# FINANCE ET JEUX RÉPÉTÉS AVEC ASYMÉTRIE D'INFORMATION

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*à la femme de ma vie, Caro*

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# Introduction

Les problèmes de gestion optimale de l'information sont omniprésents sur les marchés financiers (délit d'initié, problèmes de défaut, etc). Leurs études nécessitent une conception stratégique des interactions entre agents : les ordres placés par un agent informé influencent les cours futurs des actifs par l'information qu'ils véhiculent. Cette possibilité d'influencer les cours n'est pas envisagée par la théorie classique de la finance. Le cadre naturel de l'étude des interactions stratégiques est la théorie des jeux. Cette thèse a précisément pour objet de développer une théorie financière basée sur la théorie des jeux. Nous prendrons comme base l'article de De Meyer et Moussa Saley , *"On the origin of Brownian Motion in finance"* (**section 1.4**). Cet article modélise les interactions entre deux teneurs de marché asymétriquement informés sur le futur d'un actif risqué par un jeu répété à somme nulle à information incomplète. Cette étude montre en particulier que le mouvement Brownien, souvent utilisé en finance pour décrire la dynamique des prix, a une origine partiellement stratégique : il est introduit par les acteurs informés afin de tirer un bénéfice maximal de leur information privée. Dans la suite de cette introduction, nous détaillons la structure de cette thèse en mettant en évidence les différentes généralisations obtenues du modèle précédent.

Cette thèse est composée de 6 chapitres : Le premier rappelle le contexte des travaux dans le cadre de la microstucture des marchés financiers et de la théorie des jeux à information incomplète et les 5 derniers chapitres décrivent les résultats obtenus dans le cadre de cette thèse. La **section 1.1** présente un bref historique de la littérature actuelle sur les problèmes d'asymétrie d'information présents sur les marchés financiers. Ce survol des modèles existants nous permettra de définir un cadre d'étude approprié et de préciser les principaux objectifs à atteindre. Afin de mettre en évidence les résultats obtenus dans l'article de De Meyer et Moussa Saley, nous rappellons dans la **section 1.2** et la **section 1.3** les modèles clasiques de jeux avec manque d'information d'un côté. Nous ferons également un bref récapitulatif de l'ensemble des résultats précédemment obtenus permettant d'éclairer le lecteur sur l'apport théorique de cette thèse.
Notre étude se focalise premièrement sur le modèle de De Meyer et Moussa Saley, dont les principaux détails sont rappelés succinctement dans la **section 1.4**, et

sur des extensions naturelles. Ce modèle repose sur l'analyse d'un jeu répété avec un mécanisme de transactions très simple (enchères) ; nous remarquerons dans ce chapitre que la distribution du processus des prix est calculée explicitement, nous soulignerons le fait que cette distribution est très liée au mécanisme d'échange particulier introduit dans le modèle. Cependant, la loi limite du processus des prix, lorsque le nombre de transactions tend vers l'infini, semble indépendante de ce mécanisme. Nous envisageons d'obtenir une sorte d'universalité : obtenir la même loi limite, quel que soit le mécanisme de transaction considéré.

**Le mécanisme d'échange :**

Dans l'étude effectuée dans la section 1.4, les agents peuvent fixer des prix dans un espace continu. En réalité, sur le marché, les agents sont contraints d'annoncer des prix discrétisés. Une extension naturelle revient donc à considérer le même jeu avec des espaces d'actions finis, permettant de rapprocher le modèle d'une situation réelle et également de tester sur ce mécanisme l'universalité envisagée. Le premier objet de la thèse, exposé dans le **chapitre 2 :** ***"Continuous versus discrete market game"***, **Auteurs : B. De Meyer et A. Marino**, a été de mettre en évidence une approximation du jeu continu par le jeu discrétisé mais également d'approcher les stratégies optimales continues par les stratégies optimales discrétisées. De façon surprenante, cette étude contredit l'universalité désirée précédemment. En revanche, une analyse plus fine de ce modèle discret permet de confirmer l'apparition du mouvement brownien sur le marché financier dans un cadre plus réaliste. Cette analyse met également en évidence le comportement "optimal", induit du jeu continu, que les agents doivent adopter.

**L'asymétrie d'information :**

Le modèle de la section 1.4 considère l'interaction de deux agents asymétriquement informés. Ce manque d'information n'est analysé que dans le cadre d'une asymétrie unilatérale dans ce modèle. Pour refléter au mieux les problématiques réelles, il paraît naturel d'étendre ce modèle au cas d'une asymétrie bilatérale d'information : les agents ont une information partielle et privée sur la valeur finale d'un actif risqué. Ce modèle est détaillé dans le **chapitre 4 :** ***"Repeated market games with lack of information on both sides"*** **Auteurs : B. De Meyer et A. Marino**. Pour permettre l'étude de ce type de modèle, nous devons analyser la structure des stratégies optimales dans le cadre des jeux répétés avec manque d'information des deux côtés, dont le modèle de base et les résultats connus sont rappelés dans la **section 3.1**. L'étude de la structure récursive de ces jeux nous mène à généraliser dans la **section 3.2 :** ***"Duality and optimal strategies in the finitely repeated zero-sum games with incom-***

***plete information on both sides"* Auteurs : B. De Meyer et A. Marino** les techniques de dualité et les notions de jeu dual connus dans le cadre d'une asymétrie bilatérale d'information. Une analyse asymptotique similaire à celle effectuée dans la section 1.4 pour ce jeu fait apparaître naturellement l'étude d'un " Jeu Brownien " associé, semblable à ceux introduits dans [2]. Cette étude met également en évidence la structure du processus de prix limite. Indépendamment du fait que cette analyse apporte un résultat significatif dans le cadre des jeux financiers, elle complète, de façon théorique, l'analyse du comportement optimal des joueurs et du comportement asymptotique de la valeur d'un jeu répété à somme nulle à information incomplète des deux côtés. La théorie ne fournissant actuellement aucun résultat comparable concernant ce type de problématique.

**La diffusion de l'information :**

Une dernière extension considérée dans cette thèse concerne le processus de "diffusion de l'information". Dans une situation envisageable, les agents présents sur le marché, étant susceptibles d'acquérir de l'information, sont généralement informés progressivement et reçoivent des signaux améliorant successivement leurs connaissances privées. Nous pouvons illustrer cette intuition par l'exemple suivant : l'agent informé reçoit progressivement au cours du jeu des informations concernant l'état de santé d'une entreprise, ces signaux dépendant naturellement de l'information acquise précédemment. Ces informations sont divulguées successivement jusqu'à l'annonce du bilan annuel, correspondant à la date de révélation complète de l'information.
Nous observons que les modèles introduits dans les chapitres précédents supposent que l'information est divulguée, une fois pour toute, à l'origine du jeu au joueur informé. Nous sommes donc naturellement amenés à reconsidérer cet axiome, ainsi qu'à introduire un modèle faisant intervenir un procédé de diffusion plus général.

Nous considérerons dans cette thèse le modèle particulier dans lequel les états de la nature suivent l'évolution d'une chaîne de Markov. Les premiers résultats dans ce cadre sont dus à J.Renault dans [1], et font intervenir des jeux répétés à information incomplète d'un côté paramétrés par une chaîne de Markov.

Dans cet article, l'auteur met en évidence un premier résultat asymptotique concernant la valeur du jeu sous-jacent. La limite exhibée n'est pas exploitable sous sa forme actuelle et l'auteur souligne, sur un cas particulier très simple pour lequel aucune formule explicite n'est obtenue, la difficulté de ce type d'étude. L'objectif de la dernière partie de cette thèse a été premièrement d'élaborer un outil algorithmique permettant d'obtenir les valeurs explicites des valeurs du jeu et par là même, d'avoir une intuition sur le comportement asymptotique de celle-ci, ceci faisant l'objet du **chapitre 5 : *"An algorithm to compute the value***

***of Markov chain games"* Auteur : A. Marino**. Cet outil a également permis de résoudre le jeu régi par une chaîne de Markov particulière, donné par J.Renault dans [1], en explicitant les valeurs $V_n$ du jeu et également la limite de $\frac{V_n}{n}$, les résultats sont détaillés dans le **chapitre 6 : *"The value of a particular Markov chain game"* Auteur : A. Marino**.

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[7] Marino, A. The value of a particular Markov chain game. Chapter 6.

# Chapitre 1

# Le cadre d'étude

## 1.1 Le contexte

Dans cette section, nous esquissons un bref historique des modèles de microstucture des marchés financiers issus de l'étude de l'influence de l'asymétrie d'information. Un survol des différentes structures analysées nous permettra de mettre en évidence les avancées significatives acquises dans ce cadre d'analyse. Après une introduction rapide, nous focaliserons notre attention sur l'influence sur le marché d'une asymétrie d'information entre investisseurs. Dans une telle situation, nous rappellerons le lien évident entre le prix d'équilibre d'un actif échangé et l'influence de l'information possédée par les agents. Nous soulignerons l'efficience informationelle des prix fixés par les agents informés. La révélation de l'information se trouve être le point clef de cette étude, elle est naturellement assujettie à de nombreux facteurs tant exogènes qu'endogènes au modèle. Ces facteurs seront assimilés à des perturbations : des bruits. En prenant pour référence le modèle de Kyle [13], nous remarquerons que le modèle de De Meyer et Moussa Saley traduit de façon plus fidèle cette problématique. Ce dernier modèle mettra en particulier en évidence une origine stratégique des bruits permettant aux agents informés de tirer profit de leurs informations sans la dissimuler entièrement. Ce procédé amenuisant ainsi les capacités des agents non-informés d'inférer l'information acquise par les "insiders". Une étude plus fine du modèle nous mènera tout au long de cette thèse à en analyser différentes généralisations. En dernier lieu, nous soulignerons les avancées théoriques que peuvent dévoiler ces études.

### 1.1.1 Asymétrie d'information sur le marché financier

#### 1.1.1.1 Structure des marchés financiers

Dans cette brève introduction, nous ciblerons notre approche sur deux types de marchés : le marché de "fixing" et le marché gouverné par les prix.
Dans les marchés dits de "fixing", les agents présents sur le marché transmettent leurs ordres d'achat et de vente à un commissaire priseur, ce dernier ne prenant pas part à la transaction. La cotation et l'exécution des ordres ont lieu à intervalles de temps réguliers. Toutes les transactions se déroulent à un prix unique établit par le commissaire priseur afin d'équilibrer l'offre et la demande à la date du fixing. Sans décrire précisément les mécanismes de transaction, les ordres d'achat supérieurs au prix d'équilibre seront exécutés et inversement pour les ordres de vente. Les autres ordres ne sont pas exécutés. Ce type de mécanisme, pour lequel nous ne donnerons pas plus de détails, est fréquemment utilisé pour la détermination du prix d'ouverture des marchés.
Même si ce mécanisme de transaction apparaît comme un outil indispensable, en pratique les bourses ont le plus souvent une architecture complexe combinant plusieurs organisations distinctes. Une seconde structure employée est "le marché gouverné par les prix", celui-ci constituant la base de nos études.
Dans un marché gouverné par les prix, des investisseurs transmettent leurs ordres à un teneur de marché, "market maker", ce dernier affiche de façon continue un prix d'achat appelé "bid", et un prix de vente appelé "ask". En servant les différents ordres, le teneur de marché assure la liquidité du marché en compensant les déséquilibres éventuels entre l'offre et la demande.
Néanmoins, nous pouvons remarquer que la plupart des modèles analysant la formation des prix sur un marché gouverné par ces derniers peuvent être reformulés comme étant des marchés de fixing particuliers. Ce qui nous permet de considérer de façon théorique, notre étude telle une analyse des marchés de fixing.

Mises à part les structures intrinsèques des marchés, d'autres facteurs permettent également de les différencier : l'information, la grille des prix ...etc.

**L'information**

Nous considérons naturellement que les investisseurs présents sur le marché possèdent une information différente sur la valeur d'un actif risqué. Lors d'un échange, les prix de transaction, les quantités offertes ou demandées, révèlent une partie de l'information connue de chaque agent. L'information véhiculée influencera, après actualisation, les cours futurs de l'actif risqué sous-jacent. L'organisation du marché (règlement, transparence du marché, affichages des ordres, des prix, ...) influencera donc de façon déterminante l'efficience informationnelle de celui-ci et par là même, la valeur des cours futurs des actifs risqués. Nous analyserons en particulier dans cette thèse les positions optimales adoptées par les agents informés, afin de dévoiler le minimum d'informations.

**La grille des prix**

D'autres paramètres peuvent également influencer la liquidité et l'efficience informationnelle d'un marché. Nous introduirons et analyserons alors en particulier, l'influence de la taille de la grille des prix.
L'écart de prix entre deux ordres est en général fixé à une valeur minimale, appelée le "tick" qui varie selon les actifs, et d'un marché à l'autre. Sur le marché Américain, le "tick" est généralement de $0,125$ dollars. La taille du "tick" est un aspect de l'organisation des marchés qui est souvent débattue. Nous analyserons, dans le chapitre 2, l'influence de la grille des prix sur l'efficience informationnelle du marché et nous focaliserons également notre attention sur le choix d'un "tick" critique ou optimal.

#### 1.1.1.2 Asymétrie d'information et investisseurs

La littérature traitant de l'asymétrie d'information sur les marchés financiers est séparée en deux catégories bien distinctes. La première, initiée par Bhattacharya [1] et Ross [12], étudie l'interaction entre investisseurs, entrepreneurs ou dirigeants, dans un contexte d'asymétrie d'information. Cette analyse est principalement fondée sur la théorie du signalement, qui nécessite l'utilisation de variables telles que : les dividendes distribués par une entreprise, la part personnelle investie dans un projet ....etc. Ces variables délivrent des informations sur la valeur d'un projet proposé à l'investissement. La deuxième voie de recherche empruntée, initiée par Grossman [8], analyse principalement l'asymétrie d'information entre investisseurs. L'hypothèse principale consiste à supposer que le prix d'un titre financier est révélateur de l'asymétrie d'information existante entre les agents ayant des informations privilégiées (insiders) et les agents non informés. Dans ce cadre l'intuition est assez simple. Supposons qu'un agent dispose d'une information privée indiquant qu'un actif risqué est sous-évalué, il peut réaliser un gain immédiat en plaçant un ordre d'achat pour cet actif. L'action de l'agent informé induit un accroissement de la demande, et par là même, une augmentation des prix de l'actif risqué. Les agents non-informés peuvent déduire de cette variation que l'actif semble être sous-évalué. En interprétant correctement les signaux transmis par les ordres, les agents non-informés peuvent anticiper le lien existant entre le prix affiché et l'information de l'initié. Dans ce type de procédure, l'insider perd le bénéfice de son information pour les transactions futures. La suite de notre étude se focalise sur l'utilisation optimale de l'information acquise.

A la suite de nombreux modèles dans le cadre de la théorie des anticipations rationnelles (Grossman [8]), le point de vue adopté se situe principalement sur les problèmes de révélation de l'information au cours du temps. Les différentes études ont montré qu'il semble plus réaliste que l'agent informé minimise l'efficience informationnelle des ordres qu'il transmet. Afin de "cripter" au mieux leur information, les insiders adoptent un comportement stratégique. L'utilisation de la théorie des jeux est donc prédisposée à l'étude de ce type de problématique. L'outil principal de cette thèse sera essentiellement la théorie des jeux à information incomplète. Rappelons avant toute chose les principaux modèles existants.

**Le modèle de Kyle**

En 1985, Kyle [13] analyse la transmission de l'information par les prix dans un cadre stratégique très simple. Dans le modèle introduit, l'auteur étudie l'interaction entre trois agents asymétriquement informés. Nous considérerons de plus que le marché est constitué d'un actif risqué et d'un actif sans risque considéré comme numéraire. Les agents s'échangent les actifs, les transactions s'effectuent sur plusieurs périodes consécutives. Parmi les agents présents sur le marché, un unique agent informé apparaît ainsi que deux types d'agents non-informés : les teneurs de marché et des agents extérieurs (liquididy traders). L'asymétrie d'information se situe dans la valeur finale d'un actif risqué. La valeur finale de l'actif risqué, valeur à la fin des transactions, est supposée connue avec exactitude par l'insider ; en revanche les agents non-informés ne connaissent que sa distribution. Dans son article, Kyle considère que tous les agents sont neutres aux risques et que l'insider est l'unique agent stratégique. Les offres des liquidity traders sont supposées être des variables aléatoires exogènes, créant un bruit profitable à l'agent informé. L'information révélée par les actions de l'insider est donc masquée par ces perturbations, lui permettant de réaliser des profits aux dépends des agents non-informés. Dans ce cadre, l'efficience informationnelle des prix est diminuée. D'un autre côté, les teneurs de marché réactualisent leurs croyances comme s'ils avaient eu connaissance de la stratégie utilisée par l'insider. Or en réalité, nous remarquons que les agents non-informés ne peuvent tirer de l'information que par la quantité d'actifs demandée par l'initié. En ce sens, l'étude du comportement stratégique de l'initié proposée par Kyle semble incomplète : Comment les agents non-informés peuvent-ils réactualiser leurs croyances sans connaître la stratégie utilisée par l'initié ? Tout au plus, les agents non-informés peuvent inférer une stratégie jouée par l'initié et de ce fait, réviser leurs croyances. La plupart des modèles existants dans la littérature, en introduisant des structures de bruits exogènes, abondent dans le sens de celui de Kyle et ne mettent pas en relief l'impact du bluff dans la gestion stratégique de l'information.

**Le modèle de De Meyer et Moussa Saley**

Le modèle utilisé dans cette thèse est celui introduit par De Meyer et Moussa Saley dans [7]. Il analyse l'interaction entre deux agents asymétriquement informés échangeant un actif risqué et un actif numéraire. La structure de ces études repose sur l'étude de jeux répétés à information incomplète. Contrairement au cadre introduit par Kyle, nous supposons que les agents, informés et non-informés, ont un comportement stratégique. De Meyer et Moussa Saley fournissent explicitement les stratégies optimales des agents dans ce type de jeux, et mettent en évidence l'utilisation de perturbations par le joueur informé afin de camoufler son information privée. L'initié perturbe ses actions et bluffe l'adversaire afin d'empêcher l'agent non-informé d'inférer avec précision son information. Les stratégies optimales de l'initié dans ce cadre ne sont donc pas complètement révélatrices de son information, ce qui diminue le degré d'efficience informationnelle des prix. En dépit de toute introduction de bruits extérieurs au marché, le comportement stratégique de l'agent informé permet de retrouver l'évolution log-normale des prix. Nous remarquons que le processus de prix limite vérifie de plus une équation de diffusion semblable à celle introduite par Black et Scholes dans [2].

#### 1.1.1.3 Généralisations

Comme décrit en introduction, nous considérerons trois types de généralisation : le mécanisme d'échange, l'asymétrie d'information, la diffusion de l'information. Les différents modèles seront repris en introduction de chaque chapitre. Afin de rendre l'étude plus claire, il est nécessaire de mettre en évidence de façon théorique les jeux utilisés. La structure de la thèse sera donc naturellement un entrelacement de chapitres théoriques rappelant les résultats sur la théorie des jeux à information incomplète et de chapitres concernant les généralisations du modèle de De Meyer et Moussa Saley. L'intérêt de cette thèse ne se situe pas seulement dans la modélisation de la formation des prix sur les marchés financiers, mais également sur l'apport théorique dans le cadre des jeux à information incomplète, et plus particulièrement sur le terme d'erreur.

### 1.1.2 Terme d'erreur dans les jeux répétés avec information incomplète

Suite à l'analyse approfondie d'Aumann et Maschler du comportement asymptotique de la valeur des jeux répétés à information incomplète d'un côté, de nombreux travaux ont été effectués afin de préciser les convergences obtenues et d'affiner les résultats. Les premiers résultats concernant la vitesse de convergence de la valeur, sont dus à Mertens et Zamir dans [9]. Ces avancées théoriques ont été

obtenues dans un contexte très particulier : espaces d'actions finis et des matrices de paiements particulières. De Meyer généralisa ces résultats à un cadre plus vaste de jeux et il introduisit une notion de jeux asymptotiques appelés "jeux Brownien". Ces études ont amené De Meyer à introduire un outil permettant d'analyser les structures des stratégies optimales des joueurs : la notion de "jeu dual". La plupart des résultats obtenus n'ont pas de généralisation connue dans le cadre de manque d'information des deux côtés. Un des objectifs de cette thèse est de généraliser la notion de jeu "dual" à ce type d'environnement et par là même, de décrire la structure récursive des stratégies optimales (section 3.2). Nous verrons également apparaître, dans l'étude des jeux financiers avec asymétrie bilatérale d'information, un premier résultat concernant le terme d'erreur d'un jeu répété à information incomplète des deux côtés (chapitre 4). Cette étude asymptotique met également en évidence l'apparition d'un jeu Brownien semblable à ceux introduits dans [3].
Le deuxième apport théorique de cette thèse concerne plus précisément l'aspect intuitif. En effet, dans l'espoir d'une intuition plus précise des résultats à envisager, il apparaît nécessaire de connaître, par des méthodes algorithmiques, les valeurs d'un jeu (chapitre 5). Nous développons dans cette thèse un outil algorithmique ayant pour objectif de faciliter l'étude des "Markov chain games" introduit par J. Renault dans [11]. Cet outil permettra en particulier de résoudre explicitement un exemple non-résolu (chapitre 6).

## 1.2 Jeux à information incomplète

Dans cette section, nous présentons les principales propriétés des jeux à information incomplète d'un côté en un coup. Nous utiliserons les résultats obtenus dans cette section dans le cadre des jeux répétés à information incomplète d'un côté. Nous présenterons les résultats sous leur forme la plus générale.

### 1.2.1 Introduction et propriétés de la valeur

Soient $K$ un ensemble fini et $S$ et $T$ deux sous-ensembles convexes d'un espace vectoriel topologique. Nous définissons un jeu à somme nulle de la manière suivante : pour tout $k \in K$, nous notons $G^k$ la fonction de paiement de $S \times T$ dans $\mathbb{R}$. Le jeu procède de la manière suivante : Le joueur 1 (celui qui maximise) choisit $s$ dans $S$, le joueur 2 (celui qui minimise) choisit un élément $t$ dans $T$, le paiement du joueur 1 est alors $G^k(s,t)$.
Nous supposons de plus que $G^k$ est bilinéaire et bornée :

$$\|G\|_\infty = \sup_{k,s,t} |G^k(s,t)| < \infty$$

A chaque probabilité $p$ sur $K$, $p \in \Delta(K)$, nous associons un jeu avec manque d'information d'un côté, noté $G(p)$, qui se déroule de la manière suivante :

- A l'étape 0 : La loterie $p$ choisit un état $k$ dans $K$. Le joueur 1 est informé de $k$ mais pas le joueur 2. Le joueur 2 connaît uniquement la probabilité $p$.
- A l'étape 1 : Les joueurs choisissent simultanément une action dans leur espaces respectifs, $s$ dans $S$ et $t$ dans $T$. Le paiement est donc : $G^k(s,t)$.

Les joueurs sont informés de la description précédente du jeu. Ce dernier représenté sous forme stratégique par un triplet $(G^p, S^K, T)$. Une stratégie du joueur 1 est $s = (s^k)_{k\in K}$, où $s^k$ correspond à l'action du joueur 1 si l'état est $k$, et avec $t$ dans $T$ une stratégie du joueur 2, le paiement est

$$G^p(s,t) = \Sigma_{k\in K} p^k G^k(s^k, t)$$

Classiquement, nous noterons pour tout $p \in \Delta(K)$ :

$$\overline{v}(p) = \inf_{t\in T} \sup_{s\in S^K} G^p(s,t)$$

$$\underline{v}(p) = \sup_{s\in S^K} \inf_{t\in T} G^p(s,t)$$

Nous pouvons donc énoncé les premières propriétés des fonctions valeurs :

**Proposition 1.2.1**

1. *$\overline{v}$ et $\underline{v}$ sont Lipschitz sur $\Delta(K)$ de constante $\|G\|_\infty$.*
2. *$\overline{v}$ et $\underline{v}$ sont concaves sur $\Delta(K)$.*

Nous introduisons à présent un jeu associé, jeu dual, qui apparaîtra comme un outil très performant pour l'étude des jeux répétés avec manque d'information.

### 1.2.2 Le jeu dual

Pour $x$ dans $\mathbb{R}^n$, nous définissons le jeu dual $G^*(x)$ de la manière suivante : Le joueur 1 choisit initialement un état $k \in K$ avec la probabilité $p$, ensuite les joueurs jouent le jeu $G(p)$. Les joueurs choisissent donc $s$ dans $S$ et $t$ dans $T$, et le paiement est $x^k - G^k(s,t)$.

La forme stratégique du jeu $G^*(x)$ est la suivante : $K \times S$ est l'espace de stratégie du joueur 1 et $T$ pour le joueur 2 et la fonction de paiement définie sur $(K\times S, T)$ est : $G[x](k,s;t) := x^k - G^k(s,t)$. Contrairement au jeu primal, dans le jeu dual le joueur 1 minimise et le joueur 2 maximise.

Une stratégie mixte $\pi$ du joueur 1 est un élément de $\Delta(K \times S)$ et peut être décomposée :

$$\pi(k,s) = p^k s^k$$

où $p$ est dans $\Delta(K)$ (la marginale de $\pi$ sur $K$) et $s \in S^K$ (la distribution conditionnelle sur $S$). Nous notons donc

$$\underline{w}(x) = \sup_{t \in T} \inf_{p \in \Delta(K), s \in S^K} G[x](p, s; t)$$

$$\overline{w}(x) = \inf_{p \in \Delta(K), s \in S^K} \sup_{t \in T} G[x](p, s; t)$$

où $G[x]$ est l'extension bilinéaire de la fonction décrite en introduction.
Nous pouvons directement énoncer la propriété suivante

**Proposition 1.2.2** *$\overline{w}$ et $\underline{w}$ sont Lipschitz sur $\mathbb{R}^K$ de constante 1 et vérifient la propriété suivante : pour tout $a \in \mathbb{R}$ : $f(x + a) = f(x) + a$.*

Nous pouvons énoncer le théorème permettant de justifier la terminologie : "jeu dual". En reprenant les notations de l'appendice C, nous noterons $f^*$ la conjuguée de Fenchel de $f$.

**Proposition 1.2.3**

$$\overline{w} = (\underline{v})^* \text{ et } \underline{w} = (\overline{v})^*$$

*et par dualité*

$$\underline{v} = (\overline{w})^* \text{ et } \overline{v} = (\underline{w})^*$$

Nous pouvons donc énoncer le théorème fondamental pour la suite de notre étude

**Proposition 1.2.4** *Soit $x \in \mathbb{R}^K$, si $p$ est dans $\partial\overline{w}(x)$ et $s$ optimal pour le joueur 1 dans le jeu $G(p)$ Alors $(p, s)$ est optimal pour le joueur 1 dans le jeu $G^*(x)$.
Soit $p$ dans $\Delta(K)$, si $x$ est dans $\partial\overline{v}(p)$ et $t$ optimal pour le joueur 2 dans le jeu $G^*(x)$ Alors $t$ est optimal pour le joueur 2 dans le jeu $G(p)$.*

Ce jeu dual sera utilisé dans l'analyse des jeux répétés à information incomplète.

## 1.3 La théorie des jeux répétés avec information incomplète d'un côté

### 1.3.1 Le modèle

Nous introduisons le modèle de jeux répétés à information incomplète d'un côté sous sa forme la plus simple : avec des espaces de stratégies finis. Dans les chapitres suivants nous étudierons dans des cas particuliers ce même type de jeux, lorsque les joueurs ont des espaces continus d'actions.

Comme dans la section précédente, nous notons $G^k$ une famille de jeux, $k$ dans $K$. Par hypothèse de finitude, $G^k$ est fini et peut être identifié à une matrice $I \times J$, et $\|G\|$ devient $\max_{i,j,k} |G^k_{i,j}|$.
Pour tout $p \in \Delta(K)$, nous notons $G_n(p)$ le jeu suivant :

- A l'étape 0 : la probabilité $p$ choisit un état $k$ dans $K$, et le joueur 1 seulement est informé de $k$.
- A l'étape 1 : Le joueur 1 choisit une action $i_1 \in I$, et le joueur 2 une action $j_1 \in J$, et le couple $(i_1, j_1)$ est annoncé publiquement.
- A l'étape $q$, sachant l'histoire passée $h_{q-1} = (i_1, j_1, \ldots, i_{q-1}, j_{q-1})$, les joueurs 1 et 2 choisissent respectivement une action $i_q \in I$ et $j_q \in J$ et la nouvelle histoire $h_q = (i_1, j_1, \ldots, i_q, j_q)$ est annoncée publiquement.

Les joueurs sont informés de la description du jeu. Et nous faisons les notations suivantes :
Nous notons $H_q = (I \times J)^q$ l'ensemble des histoires à l'étape $q$ ($H_0 = \{\emptyset\}$) et $H_n = \cup_{1 \leq q \leq n} H_q$ l'ensemble de toutes les histoires. Nous notons également $S = \Delta(I)$ et $T = \Delta(J)$ les stratégies mixtes des joueurs.
Une **Stratégie Comportementale** (ou une stratégie) du joueur 1 est une application $\sigma$ de $K \times H_n$ dans $S$. Nous utiliserons la notation $\sigma = (\sigma_1, \ldots, \sigma_n)$, où $\sigma_q$ est la restriction de $\sigma$ à $K \times H_{q-1}$ : $\sigma^k_q(h_{q-1})[i]$ correspond à la probabilité que le joueur 1 joue $i$ à l'étape $q$ sachant l'histoire passée $h_{q-1}$ et l'état $k$. De façon similaire, en tenant compte de son manque d'information, une stratégie du joueur 2 est une application $\tau$ de $H_n$ vers $J$ et nous ferons également la notation $\tau = (\tau_1, \ldots, \tau_n)$. Par la suite nous noterons, $\Sigma$ et $\mathcal{T}$ les ensembles de stratégies des joueurs 1 et 2 respectivement.
Un élément $(p, \sigma, \tau)$ dans $\Delta(K) \times \Sigma \times \mathcal{T}$ induit une probabilité $\Pi_{p,\sigma,\tau}$ sur $K \times H_n$ muni de la $\sigma$-algèbre $\mathcal{K} \vee_{1 \leq q \leq n} \mathcal{H}_q$, où $\mathcal{K}$ est la $\sigma$-algèbre discrète sur $K$, et $\mathcal{H}_q$ est la $\sigma$-algèbre naturelle sur l'espace produit $H_q$.
En notant $E_{p,\sigma,\tau}$ l'espérance $E_{\Pi_{p,\sigma,\tau}}$, nous pouvons directement énoncer que

$$E_{p,\sigma,\tau} = \Sigma_{k \in K} p^k E_{k,\sigma^k,\tau}$$

où $k$ est assimilé à la masse de Dirac en $k$. Chaque séquence $(k, i_1, j_1, \ldots, i_n, j_n)$ permet d'introduire une suite de paiements $(g_q)_{1 \leq q \leq n}$ avec $g_q = G^k_{i_q, j_q}$. Le paiement du jeu est donc $\gamma^p_n(\sigma, \tau) = E_{p,\sigma,\tau}[\Sigma^n_{q=1} g_q]$. Nous remarquons que le jeu défini est un jeu fini et nous notons $V_n(p)$ sa valeur.

### 1.3.2 La martingale des aposteriori

Soit $(\sigma, \tau)$ une paire de stratégies, nous considérons la distribution induite sur $K \times H_q$ par $\Pi_{p,\sigma,\tau}$. Nous notons $p_q$ sa distribution conditionnelle sur $K$ sachant $h_q \in H_q$ : $p_q$ est la distribution aposteriori à l'étape $q$, avec $p_0 = p$. $p_q$ correspond à la croyance du joueur 2 sur l'état de la nature à l'étape $q+1$. Nous avons la propriété suivante :

**Proposition 1.3.1** *Pour tout* $(\sigma,\tau)$, $\mathbf{p} := (p_q)_{0\leq q\leq n}$ *est une* $H_q$*-martingale à valeurs dans* $\Delta(K)$. *De plus, si* $h_{q+1} \in H_{q+1}$ *:*

$$p^k_{q+1}(h_{q+1}) = p^k_q(h_q)\frac{\sigma^k(h_q)[i_{q+1}]}{\bar{\sigma}(h_q)[i_{q+1}]}$$

*avec* $\bar{\sigma}(h_q) = \Sigma_{k\in K} p^k_q(h_q)\sigma^k(h_q)$.

Nous donnons maintenant une propriété classique de cette martingale. Notons $V^1_n(\mathbf{p}) = E[\Sigma^n_{q=1}|p_q - p_{q-1}|]$ sa variation $L^1$, celle-ci sera très utile dans l'étude asymptotique de $\frac{V_n}{n}$, et nous avons directement

$$V^1_n(\mathbf{p}) \leq \sqrt{np(1-p)} \tag{1.3.1}$$

### 1.3.3 Structure récursive : Primal et Dual

La structure récursive passe par la décomposition d'un jeu de longueur $n+1$ en un jeu en 1 coup et un jeu en $n$ étapes. Nous obtenons les formules de récurrence suivantes :

**Proposition 1.3.2**

$$V_{n+1}(p) = \max_{\sigma\in S^K}\min_{\tau\in T}\left[\Sigma_{k\in K} p^k\sigma^k G^k\tau + \Sigma_{i\in I}\bar{\sigma}[i]V_n(p_1(i))\right]$$

La formule de récurrence est également vrai avec $\min\max$ au lieu de $\max\min$. La propriété précédente nous permet de conclure que : Le joueur 1 a une stratégie optimale dans $G_n(p)$ qui ne dépend, à l'étape $q$, que de $q$ et $p_{q-1}$.

Nous nous focalisons maintenant sur l'étude du jeu dual $G^*_n(x)$, $x \in \mathbb{R}^K$, du jeu $G_n(p)$, nous notons $W_n(x)$ sa valeur. $W_n$ vérifie la formule de récurrence suivante :

**Proposition 1.3.3**

$$W_{n+1}(x) = \max_{\tau\in T}\min_{i\in I} W_n(x - G_{i,\tau})$$

*où* $G_{i,\tau} = (\Sigma_{j\in J} G^k_{i,j})_{k\in K}$.

En effectuant la notation $x_q = x_{q-1} - G_{i,\tau_q}$, avec $x_0 = x$ et $\tau_q \in T$ la stratégie du joueur 2 à l'étape $q$, nous pouvons affirmer que le joueur 2 a une stratégie optimale dans $G^*_n(x)$ qui ne dépend, à l'étape $q$, que de $q$ et de $x_{q-1}$. En utilisant le résultat énoncé dans la section 1.2, donnant la relation entre les stratégies optimales du joueur 2 du primal et du dual, nous concluons que le joueur 2 a une stratégie optimale dans $G_n(p)$ ne dépendant, à l'étape $q$, que de $q$ et $(i_1, \ldots, i_{q-1})$.

Nous remarquons que les égalités précédentes ne sont en général, pas vérifiées si les espaces d'actions sont continus. Dans ce cas, l'existence de la valeur n'est pas assurée, nous obtenons donc, dans le primal, des inégalités de récurrence pour $\overline{V}_n$( maxmin du jeu) et $\underline{V}_n$ (minmax du jeu) de la forme :

$$\underline{V}_{n+1}(p) \geq \max_{\sigma \in S^K} \min_{\tau \in T} \left[\Sigma_{k\in K} p^k \sigma^k G^k \tau + \Sigma_{i\in I} \bar{\sigma}[i] \underline{V}_n(p_1(i))\right]$$
$$\overline{V}_{n+1}(p) \leq \min_{\tau \in T} \max_{\sigma \in S^K} \left[\Sigma_{k\in K} p^k \sigma^k G^k \tau + \Sigma_{i\in I} \bar{\sigma}[i] \overline{V}_n(p_1(i))\right]$$

Nous pouvons remarquer que ces inégalités permettent sous certaines conditions de prouver récursivement l'existence de la valeur. Une généralisation de ces techniques sera donnée dans le cadre d'une asymétrie bilatérale d'information, dans la section 3.2 *"Duality and optimal strategies in the finitely repeated zero-sum games with incomplete information on both sides"*.

### 1.3.4 Comportement asymptotique de $\frac{V_n}{n}$

Notons $u(p)$ la valeur du jeu précédent en 1 coup dans lequel aucun des joueurs n'a d'information privée. Un résultat général pour ce type de jeu est le suivant :

**Proposition 1.3.4**

$$cav(u)(p) \leq \frac{V_n(p)}{n} \leq cav(u)(p) + \frac{\|G\|}{n} V_n^1(\mathbf{p})$$

Ce qui nous permet de conclure en utilisant (1.3.1) que $\frac{V_n}{n}$ converge vers $cav(u)$ quand $n$ tend vers $+\infty$.

### 1.3.5 Comportement asymptotique de $\sqrt{n}(\frac{V_n}{n} - cav(u))$

Cette section approfondit l'étude en regardant la vitesse de convergence de la suite $\frac{V_n}{n}$. Nous notons, $\delta_n := \sqrt{n}(\frac{V_n}{n} - cav(u))$. Pour une classe de jeu particulier, Mertens et Zamir ont montré dans [9], que $\delta_n(p)$ converge, quand $n$ tend vers $+\infty$, vers $\phi(p) = \frac{1}{\sqrt{2\pi}} e^{-x_p^2/2}$, où $x_p$ est le $p$-quantile de la loi normale, $p = \int_{-\infty}^{x_p} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.
Mertens et Zamir ont également montré dans [10] que cette limite est reliée à l'étude asympotique de la variation $L^1$ de la martingale des aposteriori :

$$\lim_{n\to+\infty} \sup_{\mathbf{p}} \frac{V_n^1(\mathbf{p})}{\sqrt{n}} = \phi(p) \tag{1.3.2}$$

Le " sup" portant sur les $H_q$-martingale $\mathbf{p} := (p_q)_{0\leq q\leq n}$ à valeurs dans $\Delta(K)$.
Dans le jeu précédent la stratégie optimale du joueur 1 engendre donc la martingale ayant la plus grande variation $L^1$.
L'apparition de la loi normale fut expliquée, plus tardivement, par De Meyer dans [4] et [5], avec l'utilisation du jeu dual et dans [6] par une preuve directe et générale de (1.3.2). De Meyer dans [3] a également prolongé l'étude en s'intéressant à un jeu limite appelé : *Jeu Brownien*.
Dans le cadre de manque d'information des deux côtés, il n'existe pas de résultat connu permettant d'exhiber la loi normale ou le mouvement Brownien dans l'étude asymptotique de $\delta_n$. Le chapitre 4 *"Repeated market games with lack of information on both sides"* apportera une réponse à cette question dans le cas particulier des jeux financiers.
L'apparition de la loi normale dans ce type de jeu permet, en particulier dans le cadre des jeux de marché avec manque d'information d'un côté, (étudié dans [7], par De Meyer et Moussa Saley) d'apporter une explication endogène pour l'apparition du mouvement Brownien en finance. Ceci fait l'objet la section suivante.

## 1.4 Sur l'origine du mouvement Brownien en finance

B. De Meyer et H. Moussa Saley

Nous donnons dans cette section un rappel succinct des résultats obtenus par De Meyer et Moussa Saley dans le modèle de jeux financiers avec manque d'information d'un côté. La description de ce modèle sera approfondie dans les chapitres 2 et 4 : *"Continuous versus discrete market game"* et *"Repeated market games with lack of information on both sides"* .

### 1.4.1 Le modèle

Dans ce jeu, nous supposons que les espaces d'actions sont continus, $I = J = [0,1]$, et que l'ensemble des états de la nature est $K := \{H, L\}$. Dans la suite nous assimilerons le simplex $\Delta(K)$ à l'intervalle $[0,1]$. Nous définissons la fonction de paiement par le mécanisme d'échange à chaque étape. Si le joueur 1 fixe $p_{1,q}$ et le joueur 2 fixe $p_{2,q}$ à l'étape $q \in \{1,\dots,n\}$, nous avons $g_q(k, p_{1,q}, p_{2,q})$ est égal à

$$g_q(k, p_{1,q}, p_{2,q}) = \mathbb{1}_{p_{1,q}>p_{2,q}}(\mathbb{1}_{k=H} - p_{1,q}) + \mathbb{1}_{p_{1,q}<p_{2,q}}(p_{2,q} - \mathbb{1}_{k=H})$$

Nous notons dans la suite $G_n(p)$ le jeu répété avec manque d'information d'un côté associé. Sachant que les espaces d'actions considérés sont continus, les résultats obtenus précédemment concernant l'existence de la valeur ne peuvent pas s'appliquer.

### 1.4.2 Les principaux résultats

**Proposition 1.4.1** *Le jeu $G_n(p)$ a une valeur $V_n(p)$ et les joueurs ont des stratégies optimales. $V_n(p)$ est concave sur $[0,1]$.*

Cet article donne de plus une formule explicite de la valeur. Pour cela, notons $f_n$ la densité de la variable aléatoire $S_n := \Sigma_{q=1}^n \frac{U_q}{\sqrt{n}}$, où $U_1, \ldots, U_n$ sont des variables i.i.d uniformes sur $[-1,1]$. Nous obtenons

**Proposition 1.4.2** *Pour tout $p$ dans $[0,1]$,*

$$\frac{V_n}{\sqrt{n}}(p) = \int_{x_p^n}^{+\infty} s f_n(s) ds$$

*Où $x_p^n$ vérifie $p = \int_{x_p^n}^{+\infty} f_n(s) ds$.*

Nous remarquons que la symétrie du problème implique directement dans ce cas, que la valeur $u$ est nulle ( $\Rightarrow cav(u) = 0$). Ce qui légitime l'étude asymptotique de $\frac{V_n}{\sqrt{n}}$. Nous savons par le théorème central limit que $f_n(x)$ converge vers $\sqrt{3} f(\sqrt{3}x)$, où $f$ est la densité de la loi normale : $f(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$. Ceci permet directement d'énoncer comme corollaire

**Proposition 1.4.3** *$\sqrt{3}\frac{V_n}{\sqrt{n}}(p)$ converge, quand $n$ tend vers $+\infty$, vers $f(z_p)$, où $z_p$ vérifie $p = \int_{z_p}^{+\infty} f(s) ds$.*

Ce jeu avec espaces d'actions infinis apporte une généralisation des résultats obtenus dans la section précédente. Dans ce cas précis, nous pouvons donner également une description précise des stratégies optimales des joueurs. Nous connaissons donc la distribution du processus de prix proposés : $\{(p_{1,q}^n, p_{1,q}^n)\}_{q=1,\ldots,n}$, où $p_{i,q}^n$ correspond au prix fixé par le joueur $i$ à l'étape $q$ dans le jeu de longueur $n$. Le processus de prix de transaction devient donc $\{p_q^n\}_{q=1,\ldots,n}$, avec $p_q^n = \max(p_{1,q}^n, p_{2,q}^n)$. En représentant le processus $p^n$ par un processus continu $\pi^n$ : $\pi_t^n := p_q^n$ si $t \in [\frac{q-1}{n}, \frac{q}{n}[$, nous obtenons le résultat suivant

**Proposition 1.4.4** *Le processus $\pi^n$ converge en loi, quand $n$ tend vers $+\infty$, vers le processus $\pi$ vérifiant :*

$$\pi_t := F\left(\frac{z_p + B_t}{\sqrt{1-t}}\right)$$

*Avec, $B$ un mouvement Brownien standard, $B_0 = 0$, et $F(x) = \int_x^\infty f(s) ds$.*
*Le processus $\pi$ est une martingale continue à valeurs dans $[0,1]$ tel que $\pi_0 = p$ et $\pi_1$ appartient presque sûrement à $\{0,1\}$.*

Nous remarquons que l'application du lemme d'Ito à la formule donnant $\pi$ fait apparaître une équation de diffusion pour le processus de prix semblable à celle introduite par Black-Scholes dans [2].
Le modèle proposé dans cette section n'est pas complètement réaliste, mais il est un premier pas permettant de mettre en évidence de l'origine partiellement stratégique du mouvement Brownien en finance.

### 1.4.3 Extensions possibles

Différentes extensions sont envisageables dans ce type de modélisations : mécanisme d'échange, asymétrie bilatérale d'information ...etc.

#### Le cas discret

La première remarque que nous pouvons effectuer concerne l'hypothèse de la continuité des espaces de prix. Nous observons que les prix sur les marchés financiers sont généralement fixés avec un nombre déterminé de chiffres significatifs (ex : 4). Cette notion est liée au choix de la taille (le ***tick***) de la grille des prix disponibles. Nous pouvons donc considérer le jeu précédent en supposant de plus que les joueurs sont contraints de choisir leurs prix dans un espace discret. Les espaces de stratégies deviennent une discrétisation régulière de l'intervalle $[0, 1]$ : $\{i\delta | i = 0, \ldots, \frac{1}{\delta}\}$, $\delta$ correspondant au pas de la discrétisation. Les questions sous-jacentes sont les suivantes :

- La loi normale apparaît-elle dans le comportement asymptotique de la valeur ?
- Peut-on confirmer l'apparition du mouvement Brownien ?
- Le jeu continu est-il une bonne approximation du discrétisé ?

Les réponses à ces questions font l'objet du chapitre suivant : *"Continuous versus discret market games"*.

#### L'asymétrie bilatérale d'information

Il est également naturel de considérer que l'information sur la valeur finale de l'actif est plus fréquemment partagée entre les agents. Nous pouvons donc considérer le modèle allouant initialement une information partielle et privée à chacun des deux agents. L'étude de ce modèle est directement liée à la théorie des jeux répétés à information incomplète des deux côtés. A la suite d'une introduction des résultats connus sur ce type de jeu dans la section 3.1, Le chapitre 4 *"Repeated* *market games with lack of information on both sides* " traitera de cet extension. En particulier ce chapitre répondra aux questions suivantes :

- La loi normale apparaît-elle dans le comportement asymptotique de la valeur divisée par $\sqrt{n}$ ?
- A-t-on une formule explicite de la valeur et des stratégies optimales ?
- Dans le cas contraire, pouvons nous fournir une valeur asymptotique mettant en évidence le mouvement Brownien ?

**La diffusion de l'information**

Les modèles précédents considèrent que l'information est fournie, une fois pour toute, à l'origine du jeu. Si par exemple, nous considérons qu'un agent est informé progressivement au cours du jeu de l'état de santé d'une entreprise, les modèles doivent faire intervenir un processus de diffusion de l'information. Le premier modèle étudié dans cette thèse est celui considérant que l'état de la nature suit l'évolution d'une chaîne de Markov. Les premiers résultats dans ce cadre sont dus à J.Renault dans [11], dans cet article l'auteur prouve, en particulier, l'existence de la limite de $\frac{V_n}{n}$. La limite exhibée n'est pas exploitable sous sa forme actuelle, l'auteur met également en évidence un cas particulier très simple pour lequel les valeurs et la limite ne sont pas connues. L'objectif de la dernière partie de cette thèse a été premièrement d'élaborer un outils algorithmique permettant d'obtenir les valeurs explicites $V_n$ et par là même, d'avoir une intuition sur la limite de $\frac{V_n}{n}$, ceci fait l'objet du chapitre 5 *"An algorithm to compute the value of a Markov chain game* ". Cet outil a également permis de résoudre le jeu régi par une chaîne de Markov particulière, donné par J.Renault dans [11] en explicitant les valeurs $V_n$ et également la limite de $\frac{V_n}{n}$. Les résultats sont démontrés dans le chapitre 6 " *The value of a particular Markov chain game*".

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# Chapitre 2

# Continuous versus discrete market games

B. De Meyer and A. Marino

De Meyer and Moussa Saley [4] provide an endogenous justification for the appearance of Brownian Motion in Finance by modeling the strategic interaction between two asymmetrically informed market makers with a zero-sum repeated game with one-sided information. The crucial point of this justification is the appearance of the normal distribution in the asymptotic behavior of $\frac{V_n(P)}{\sqrt{n}}$. In De Meyer and Moussa Saley's model [4], agents can fix a price in a continuous space. In the real world, the market compels the agents to post prices in a discrete set. The previous remark raises the following question : " Does the normal still appear in the asymptotic of $\frac{V_n}{\sqrt{n}}$ for the discrete market game ? ". The main topic is to prove that for all discretization of the set price, $\frac{V_n(P)}{\sqrt{n}}$ converges uniformly to 0. Despite of this fact, we don't reject De Meyer, Moussa analysis : when the size of the discretization step is small as compared to $n^{-\frac{1}{2}}$, the continuous market game is a good approximation of the discrete one.

## 2.1 Introduction

Financial models of the price dynamic on the stock market often incorporate a Brownian term (see for instance Black and Scholes [3]). This Brownian term is often explained exogenously in the literature : the price of an asset depends on a very long list of parameters which are subject to infinitesimal random variations with time (as for instance the demographic parameters). Due to an aggregation result in the spirit of the Central Limit theorem, these variations are responsible for the Brownian term in the price dynamic. However, this kind of explanation does not apply to discontinuous parameters that are quite frequent in the real

world. For instance, the technological index of a firm will typically jump whenever a new production process is discovered. With the above exogenous explanation, such a discontinuity of the parameter process (a shock) would automatically generate a discontinuity of the price process. In [4], De Meyer and Moussa Saley provide an endogenous justification for the appearance of the Brownian term even in case of discontinuous parameters. They also explain how the market will preserve the continuity of the price process. Their explanation is based on the informational asymmetries on the market. When such a shock happens, some agent are informed and others are not. At each transaction, the optimal behavior of the informed agents will be a compromise between an intensive use of his information at that period and a constant concern of preserving his informational advantage for the next periods. To obtain this compromise, the insiders will slightly noise their actions day after day and asymptotically these noises will aggregate in a Brownian Motion.
To support this thesis, De Meyer and Moussa Saley analyze the interaction between two asymmetrically informed market makers : Two market makers, player 1 and 2, are trading two commodities N and R. Commodity N is used as numéraire and has a final value of 1. Commodities R( R for risky asset ) has a final value depending on the state $k$ of nature $k \in K := \{L, H\}$. The final value of commodity R is 0 in state $L$ and 1 in state $H$. By final value of an asset, we mean its liquidation price at a fixed horizon T, when the state of nature will be publicly known.
The state of nature $k$ is initially chosen at random once for all. The probability of $H$ and $L$ being respectively $P$ and $1 - P$. Both players are aware of this probability. Player 1 is informed of the resulting state $k$ while player 2 is not.

The transactions between the players, up to date T, take place during $n$ consecutive rounds. At round $q$ ( $q = 1, \ldots, n$ ), player 1 and 2 propose simultaneously a price $p_{1,q} \in D$ and $p_{2,q} \in D$ for 1 unit of commodity R ( $D \subset \mathbb{R}$ ). The maximal bid wins and one unit of commodity R is transacted at this price. If both bids are equal, no transaction happens.

In other words, if $y_q = (y_q^R, y_q^N)$ denotes player 1's portfolio after round $q$, we have $y_q = y_{q-1} + t(p_{1,q}, p_{2,q})$, with

$$t(p_{1,q}, p_{2,q}) := \mathbb{1}_{p_{1,q} > p_{2,q}}(1, -p_{1,q}) + \mathbb{1}_{p_{1,q} < p_{2,q}}(-1, p_{2,q}).$$

The function $\mathbb{1}_{p_{1,q} > p_{2,q}}$ takes the value 1 if $_{p_{1,q} > p_{2,q}}$ and 0 otherwise.

At each round the players are supposed to remind the previous bids including those of their opponent. The final value of player 1's portfolio $y_n$ is then $\mathbb{1}_{k=H} y_R + y_N$. We consider the players are risk neutral, so that the utility of the players is the expectation of the final value of their own final portfolio. There is no loss of generality to assume that initial portfolios are $(0,0)$ for both players. With that assumption, the game $G_n^D(P)$ thus described is a zero-sum repeated game with one-sided information as introduced by Aumann and Maschler [1].

As indicated above, the informed player will introduce a noise on his actions. Therefore, the notion of strategy we have in mind here is that of behavior strategy. More precisely, a strategy $\sigma$ of player 1 in $G_n^D(P)$ is a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_q$ is the lottery on $D$ used by player 1 at stage $q$ to selects his price$p_{1,q}$. This lottery will depend on player 1's information at that stage which includes the states as well as both player's past moves. Therefore $\sigma_q$ is a (measurable) mapping from $\{H, L\} \times D^{q-1}$ to the set $\Delta(D)$ of probabilities on $D$. In the same way, a strategy $\tau$ of player 2 is a sequence $(\tau_1, \ldots, \tau_n)$ such that $\tau_q : D^{q-1} \rightarrow \Delta(D)$.

A pair of strategies $(\sigma, \tau)$ joint to $P$ induces a unique probability $\Pi_{P,\sigma,\tau}$ on the histories $k \in \{H, L\}, p_{1,1}, p_{2,1}, \ldots, p_{1,n}, p_{2,n}$. The payoff $g(P, \sigma, \tau)$ in $G_n^D(P)$ corresponding to the pair of strategy $(\sigma, \tau)$ is then $E_{\Pi_{P,\sigma,\tau}}[\mathbb{1}_{k=H} y_R + y_N]$.
The maximal amount player 1 can guarantee in $G_n^D(P)$ is

$$\underline{V}_n^D(P) := \sup_\sigma \inf_\tau g(P, \sigma, \tau)$$

and the minimal amount player 2 can guarantee not to pay more is $\overline{V}_n^D(P) := \inf_\tau \sup_\sigma g(P, \sigma, \tau)$. If both quantities coincide the game is said to have a value. A strategy $\sigma$ (resp. $\tau$) such that $\underline{V}_n^D(P) = \inf_\tau g(P, \sigma, \tau)$ (resp. $\overline{V}_n^D(P) := \sup_\sigma g(P, \sigma, \tau)$ is said to be optimal.

Before dealing with the main topic of this paper, let us discuss the economical interpretation of this model. A first observation concerns the fact that the model is a zero sum game with positive value : This means in particular that the uninformed market maker will lose money in this game, so, why should he take part to this game ? To answer this objection, we argue that, once an institutional has agreed to be a market maker, he is committed to do so. The only possibility for him not to participate to the market would be by posting prices with a huge bid-ask spread. However, there are rules on the market that limit drastically the allowed spreads. In this model the spread is considered as null since the unique price posted by a player is both a bid and an ask price. The above model has to be considered as the game between two agents that already have signed as Market Makers, one of which receives after this some private information.

The second remark we would like to do here is on the transaction rule : The price posted by a Market Maker commits him only for a limited amount : when a bigger number of shares is traded, the transaction happens at the negotiated price which is not the publicly posted price. We suppose in this model that the

price posted by a Market Maker only commits him for one share.
Now, if two market makers post a prices that are different, say $p_1 > p_2$, there will clearly be a trader that will take advantage of the situation : The trader will buy the maximal amount (one share) at the lowest price ($p_2$) and sell it to the other market maker at price $p_1$. So, if $p_1 > p_2$, one share of the risky asset goes from market maker 1 to market maker 2, and this is indeed what happens in the above model. The above remark also entails that each market maker trades the share at his own price in numéraire. This is not taken into account in De Meyer Moussa Saley model, since the transaction happens there for both market makers at the maximal price. Introducing this in the model would make the analysis much more difficult : the game would not be zero sum any more, and all the duality techniques used in [4] would not apply. The analysis of a model with non zero sum transaction rules goes beyond the scope of this paper, but will hopefully be the subject of a forthcoming publication.

De Meyer- Moussa Saley were dealing with the particular case $D = [0, 1]$ and the corresponding game will be denoted here $G_n^c(P)$ ($c$ for continuous) and their main results, including the appearance of the Brownian motion, are reminded in the next section.

It is assumed in $G_n^c$ that the prices posted by the market makers are any real numbers in [0,1]. In the real world however, market makers are committed to use only a limited numbers of digits, typically four. In this paper, we are concerned with the same model but under the additional requirement that the prices belong to some discrete set : we will also consider the discretized game $G_n^l(P) := G_n^{D_l}(P)$ where $D_l := \{\frac{i}{l-1}, i = 0, \ldots, l-1\}$. The main topic of this paper is the analysis of the effects of this discretization.

As we will see, the discretized game is quite different from the continuous one : It is much more costly to noise his prices for the informed agent in $G_n^l$ than in $G_n^c$ : he must use lotteries on prices that differ at least by the tick $\delta := \frac{1}{l-1}$ while in $G_n^c$, the optimal strategies are lotteries whose support is asymptotically very small (and thus smaller than $\delta$).

The question we address in this paper is the following : As $n \to \infty$, does the Brownian motion appear in the asymptotic dynamics of the price process for the discretized game ?
As we will see in section 3, the answer is negative. At first sight, this result questions the validity of De Meyer, Moussa's analysis. We compare therefore in section 5 the discrete game with the continuous one. In particular, we show that the continuous model remains a good approximation of the discrete one, as far as $\sqrt{n}\delta$ is small, where $\delta$ is the discretization step and $n$ is the number of transactions. When this is the case, we prove that discretizing the optimal strategies of the continuous game provides good strategies for $G_n^l$. The fact that $\sqrt{n}\delta$ is small in general explains why the analysis made in [4] remains valid.

## 2.2 Reminder on the continuous game $G_n^c$

De Meyer, Moussa Saley prove in [4] that the game $G_n^c(P)$ has a value $V_n^c(P)$. Furthermore, they provide explicit optimal strategies for both players.

The keystone of their analysis is the recursive structure of the game, and a new parametrization of the first stage strategy spaces. Namely, at the first stage, player 1 selects a lottery $\sigma_1$ on the first price $p_1$ he will post, lottery depending on his information $k \in \{H, L\}$. In fact, his strategy may be viewed as a probability distributions $\pi$ on $(k, p_1)$ satisfying : $\pi[k = H] = P$.
In turn, such a probability $\pi$ may be represented as a pair of functions $(f, Q)$ ($[0,1] \to [0,1]$) satisfying :

$$\begin{array}{ll} (1) & f \text{ is increasing} \\ (2) & \int_0^1 Q(u)du = P \\ (3) & \forall x, y \in [0,1] : f(x) = f(y) \Rightarrow Q(x) = Q(y) \end{array} \tag{2.2.1}$$

The set of these pairs will be denoted by $\Gamma_1^c(P)$ in the sequel.
Given such a pair $(f, Q)$, player 1 generates the probability $\pi$ as follows : he first selects a random number $u$ uniformly distributed on $[0,1]$, he plays then $p_1 := f(u)$ and he then chooses $k \in K$ at random with a lottery such that $p[k = H] = Q(u)$.

In the same way, the first stage strategy of player 2 is a probability distribution for $p_2 \in [0,1]$. To pick $p_2$ at random, player 2 may proceed as follows : given a increasing function $h$ :$[0,1] \to [0,1]$, he selects a random number $u$ uniformly distributed on $[0,1]$ and he plays $p_2 = h(u)$. Any distribution can be generate in this way and therefore we may identify the strategy space of player 2 with set $\Gamma_2^c$ of these functions $h$.

Based on that representation of player 1 first stage strategies, the recursive formula for $V_n^c$ becomes :

**Theorem 2.2.1** *[The primal recursive formula]*

$$V_{n+1}^c = T^c(V_n^c),$$

*where*

$$T^c(g)(P) = sup_{(f,Q)\in\Gamma_1^c(P)} inf_{p_2\in[0,1]} F((f, Q), p_2, g),$$

*with*

$$F((f, Q), p_2, g) := \int_0^1 \{1\!\!1_{f(u)>p_2}(Q(u) - f(u)) + 1\!\!1_{f(u)<p_2}(p_2 - Q(u))\}du + \int_0^1 g(Q(u))du$$

A first move optimal strategy $\sigma_1$ in $G^c_{n+1}(P)$ for player 1 corresponds to a pair $(f^\circ, Q^\circ)$ which verifies :

$$V^c_{n+1}(P) = inf_{p_2\in[0,1]} F((f^\circ, Q^\circ), p_2, V^c_n).$$

After the first stage, player 1 plays optimally in $G^c_n(Q(u))$.

Another useful tool in De Meyer, Moussa Saley analysis is Fenchel duality : it is quite natural to use it in this framework since $V^c_n$ is proved to be concave.

**Definition 2.2.2** *the Fenchel conjugate $f^*$ (or simply conjugate ) of $f$ is defined as follows : $f^* : \mathbb{R} \to [-\infty, +\infty)$ such that :*

$$f^*(x) = inf_{P\in[0,1]} xP - f(P)$$

From this definition, it is obvious that :

$$\text{If } f \leq g \text{ then } g^* \leq f^* \tag{2.2.2}$$

The Fenchel conjugate $W^c_n := (V^c_n)^*$ of $V^c_n$ may be interpreted as the value of a dual game. The recursive structure of this dual game is particularly well suited to analyze the optimal strategies of player 2.

**Theorem 2.2.3** *[The dual recursive formula] For all $x \in \mathbb{R}$ :*

$$W^c_{n+1}(x) = \Lambda^c(W^c_n)(x),$$

*where $\Lambda^c(g)(x) = sup_{h\in\Gamma^c_2} inf_{p_1\in[0,1]} R[x](p_1, h, g)$,*
*with*

$$R[x](p_1, h, g) := g(x - \int_0^1 1\!\!1_{h(u)<p_1} - 1\!\!1_{h(u)>p_1} du) - \int_0^1 1\!\!1_{h(u)<p_1}(-p_1) + 1\!\!1_{h(u)>p_1} h(u) du$$

An optimal strategy for player 2's is a function $h^\circ$ which verifies :

$$W^c_{n+1}(x) = inf_{p_1\in[0,1]} R[x](p_1, h^\circ, W^c_n)$$

The following Formulas, corresponding to the formula (6) and (8) in [4], provide explicit optimal strategies for player 1 in $G^c_n(P)$. For all $u \in [0,1]$

$$\begin{array}{rcl} u^2 f(u) & = & \int_0^u 2sQ(s)ds \\ Q(u) & = & (W^c_n)'(\lambda + 1 - 2u) \end{array}$$

where $(W^c_n)'$ is the derivative of the function $W^c_n$ and $\lambda$ is such that the expectation of $Q$ is equal to $P$. Explicit expression for optimal $h^*$ is given in formula (20) in [4] which is : for all $u \in [0,1]$

$$h(u) = u^{-2} \int_0^u 2s(W^c_n)'(x - 2s + 1)ds$$

The main result of [4] is the appearance of Brownian Motion in the asymptotic dynamic of the price process in $G_n^c(P)$ as $n$ goes to infinity : Since optimal strategy of players are explicitly known, we may compute the distribution of the proposed price process of player 1 $(p_{1,1}^n, \dots, p_{1,n}^n)$ in $G_n^c(P)$. This process $p_1^n$ may be viewed as a continuous time process $\Pi^n$ on $[0,1]$ with $\Pi_t^n = p_{1,q}^n \; if \; \frac{q}{n} \leq t < \frac{q+1}{n}$.
With the previous notation, De Meyer and Moussa Saley (see [4]) prove the following asymptotic result :

**Theorem 2.2.4** *As $n$ goes to $\infty$ , the process $\Pi_t^n$ converges in law, in the sense of finite distribution, to the following process $\Pi$ :*

$$\Pi_t = F(\frac{z_p + B_t}{\sqrt{1-t}})$$

*Where $F(x) = \int_x^\infty f(z)dz$ , $z_p$ such that $F(z_p) = p$ and $B_t$ is a M.B. The process $\Pi$ is a $[0,1]$-valued continuous martingale starting at $P$ at time $0$. Furthermore $\Pi_1$ belongs almost surely to $\{0,1\}$.*

This result is in fact related to the following one :

**Theorem 2.2.5** *Let $f$ the normal density : $f(z) := exp(-\frac{z^2}{2})/\sqrt{2\pi}$.
As $n$ goes to $\infty$, $\Psi_n^c(P) := \frac{V_n^c}{\sqrt{n}}(P)$ converges to $\frac{1}{\sqrt{3}} f(z_P)$, where $z_P$ is such that $P = \int_{z_P}^\infty f(s)ds$.*

In the next section, we prove that the value $V_n^l(P)$ of the discretized game doesn't have the same asymptotic as $V_n^c(P)$. There is therefore no hope for the appearance of a Brownian Motion in the dynamic of the discretized price process. This phenomena could heuristically be explained as follows.

From theorem 15 and lemma 9 in [4], there exists a constant $C$ such that for all $n, m$, with $m < n$ :

$$|p_{1,m}^n - p_{2,m}^n| \leq C/\sqrt{n-m},$$

where $p_{i,m}^n$ is the price posted by player $i$ in the $m$'th stage of $G_n^c(P)$.

Therefore, once $C/\sqrt{n-m}$ is less than the discretization step $\frac{1}{l-1}$ the players should post the same price. Due to the transaction rules, this means a zero payoff for both players in the beginning of the game. This will be true as far as $m \leq n - ((l-1)C)^2$, so only $((l-1)C)^2$ transactions could give a positive payoff ( smaller than 1) to player 1 : the value of the discrete market game would be bounded by $((l-1)C)^2$. This is the content of theorem 2.3.1.

## 2.3 The discretized game $G_n^l$

In this section we are concerned with the game $G_n^l := G_n^{D_l}$ where $D_l := \{\frac{i}{l-1}, i = 0, \ldots, l-1\}$.

This game is in fact a standard repeated game as introduced in Aumann Mashler with $D_l$ as action set and with $A^H$, $A^L$ as payoff matrices :

$$A^H = \begin{pmatrix} 0 & \delta - 1 & ... & i\delta - 1 & (i+1)\delta - 1 & ... & 0 \\ 1-\delta & 0 & ... & ... & ... & ... & 0 \\ ... & ... & 0 & i\delta - 1 & ... & ... & ... \\ 1 - i\delta & ... & 1 - i\delta & 0 & (i+1)\delta - 1 & ... & ... \\ 1-(i+1)\delta & ... & ... & 1-(i+1)\delta & 0 & ... & ... \\ ... & ... & ... & ... & ... & 0 & 0 \\ 0 & ... & ... & ... & ... & 0 & 0 \end{pmatrix}$$

and :

$$A^L = \begin{pmatrix} 0 & \delta & ... & i\delta & (i+1)\delta & ... & 1 \\ -\delta & 0 & ... & ... & ... & ... & 1 \\ ... & ... & 0 & i\delta & ... & ... & ... \\ -i\delta & ... & -i\delta & 0 & (i+1)\delta & ... & ... \\ -(i+1)\delta & ... & ... & -(i+1)\delta & 0 & ... & ... \\ ... & ... & ... & ... & ... & 0 & 1 \\ -1 & ... & ... & ... & ... & -1 & 0 \end{pmatrix}$$

(Line $i$ corresponds to price $p_1 = i\delta$ with $\delta = \frac{1}{l-1}$, and similarly for column $j$.)

From Aumann and Maschler's paper, the game $G_n^l(P)$ has a value hereafter denoted by $V_n^l(P)$ and both players have optimal strategies.

The next section is devoted to the proof of the next theorem :

**Theorem 2.3.1** *For $n = 0, 1, \ldots$, for all $P \in [0,1]$, $V_n^l(P)$ is an increasing sequence in $n$ with limit $g^l(P)$, where $g^l(P)$ is linear for $P$ in $[\frac{k}{l-1}, \frac{k+1}{l-1}]$ for each $k$ in $\{0, \ldots, l-2\}$ and such that for $P$ in $D_l$, $g^l(P) := P(1-P)\frac{1}{2\delta}$.*

The proof of this theorem is based on the well known recursive structure of the Aumann and Maschler repeated games that expresses $V_{n+1}^l$ as $T(V_n^l)$ where $T$ is the following recursive operator :

$$T(g)(P) = \max_{(\sigma_H, \sigma_L)} \min_{\tau} [\sum_{k \in \{H,L\}} P^k \sigma_k A^k \tau + \sum_{i=1}^{l} \sigma(i) g(P(i))] \tag{2.3.1}$$

with $\sigma = P\sigma_H + (1-P)\sigma_L$ and, if $\sigma(i) > 0$, $P(i) = \frac{P\sigma_H(i)}{\sigma(i)}$.
The pair $(\sigma_H, \sigma_L)$ joint to $P$ induces a probability distribution on $K \times D_l$ which in turn can be represented by its marginal distribution $\sigma$ on $D_l$ and by $P(.)$, where $P(i)$ is as above the conditional probability of $H$ given $i$. In particular we have $E_\sigma[P(i)] = P$. In this framework, $T$ may be written as :

$$T(g)(P) = \max_{\{(\sigma(i),P(i))\ st\ E_\sigma[P(i)]=P\}} [\min_j (\sum_{i=1}^{l} \sigma(i)[1\!\!1_{i>j}(P(i)-i\delta)+1\!\!1_{i<j}(j\delta-P(i))+g(P(i))])] \tag{2.3.2}$$

To play optimally in $G_n^l(P)$, player 1 proceeds as follows : At the first stage, he plays $\sigma_H$ and $\sigma_L$ optimal in $T(V_{n-1}^l)(P)$ and he then computes the a posteriori $P^1(i_1) := P(i_1)$. From there on, he plays optimally in $G_{n-1}^l(P^1(i_1))$. In particular, he plays at the second stage an optimal move in $T(V_{n-2}^l)(P^1(i_1))$. He then computes the a posteriori probability $P^2(i_1, i_2)$ of $H$ and plays for the remaining stages an optimal strategy in $G_{n-2}^l(P^2(i_1, i_2))$. So that the a posteriori martingale $P^1, \ldots, P^n$ may be viewed as a stage variable for player 1 : at stage $q$, he just has to remind $P^q$ to play optimally in $G_n^l(P)$.

The fact that $V_n^l$ is increasing in $n$ just results from the fact that for all concave continuous function $V$, $V \le T(V)$ (see lemma 2.4.2).

We then have to prove that $V_n^l$ is bounded from above by $g^l$. Since $T$ is an increasing operator (if $h \le g$ then $T(h) \le T(g)$), a positive fixed point $g$ for operator $T$ will be an upper bound for $V_n^l$ (see lemma 2.4.3). We have then to find such a fixed point, but the operator $T$ is a bit complicated to analyze directly so we introduce an operator $T^*$ that dominates $T$ (for all $V$, $T(V) \le T^*(V)$) for which we prove that $g^l$ is a fixed point and therefore also a fixed point for $T$ (see lemma 2.4.4).
It then remains to prove the convergence of $V_n^l$ to $g^l$ and this is obtained as follows : Since we suspect that for high $n$, $V_n^l$ should be close to $g^l$, the optimal strategy in $T(V_n^l)$ should be close to an optimal strategy in $T(g^l)$. We then consider a strategy $\sigma^{n,l}$ of player 1 in $G_n^l(P)$ that consists at stage $q$ in playing the optimal strategy in $T(g^l)(P^q)$, where $P^q$ is the a-posteriori after stage $q$. The amount $C_n^l(P)$ guaranteed by that strategy in $G_n^l(P)$ is clearly a lower bound of $V_n^l(P)$.

We next prove that $C_n^l$ converges to $g^l$ as follows :
When $P$ belongs to $D_l \backslash \{0,1\}$, we prove in theorem 2.4.11 that the following strategy $(\sigma_H, \sigma_L)$ is optimal in $T(g^l)(P)$ : let $P^+ := P + \delta$ and $P^- := P - \delta$. Both $\sigma_H$ and $\sigma_L$ are lotteries on the prices $P$ and $P^-$ with $\sigma_H(P) = \frac{P^+}{2P}$ and $\sigma_L(P) = \frac{1-P^+}{2(1-P)}$. With such a strategy, player 1 plays $P$ with probability

$P\sigma_H(P) + (1-P)\sigma_L(P) = \frac{1}{2}$ and therefore $P^1(P)$ is equal to $2P\sigma_H(P) = P^+$. Similarly player 1 plays $P^-$ with probability $P\sigma_H(P^-) + (1-P)\sigma_L(P^-) = \frac{1}{2}$ and therefore $P^1(P^-)$ is equal to $2P\sigma_H(P^-) = P^-$. Therefore, with that strategy the a posteriori $P^1$ and the price posted by player 1 differ at most by $\delta$. Furthermore, the a posteriori belongs clearly to $D_l$.

The price process induces by the strategy $\sigma^{n,l}$ remains at most at a distance $\delta$ of the a posteriori martingale $(P^q)_{q=1,\dots,n}$. If $P^q$ is in $]0,1[$, then $P^{q+1}$ is equal to $P^{q,+}$ or $P^{q,-}$, each with probability $\frac{1}{2}$. Furthermore, if $P^q$ is equal to 0 or 1 then $P^{q+1} = P^q$ and price fixed by player 1 are respectively 0 and 1. So, the process $(P^q)_{q=1,\dots,n}$ is a $D_l$-valued symmetric random walk stopped at the time $\tau$ when it reaches 0 or 1.

As proved in theorem 2.4.11, the best reply of player 2 against $\sigma^{n,l}$ is to post at stage $q$ a price equal to $P^{q-1}$. So, this allows us to compute explicitly $C_n^l$. At stage $q$, player 1 get exactly

$$E[1\!\!1_{p_1 > P^{q-1}}(P^q - p_1) + 1\!\!1_{p_1 < P^{q-1}}(P^{q-1} - P^q)]$$

The price posted by player 1 is either $P^{q-1}$ or $P^{q-1} - \delta$, so the first term is always equal to 0. The second term takes only the value $\delta$ when the price posted by player 1 is $P^{q-1} - \delta$ which happens with probability $\frac{1}{2}$. Hence, the expectation is just $\frac{\delta}{2}$, if $P^{q-1}$ is not equal to 0 or 1. In case $P^{q-1} = 0$ or 1, the previous expectation is equal to 0. As a consequence, $C_n^l$ is just equal to :

$$C_n^l = E[\sum_{q=1}^{n} 1\!\!1_{q \le \tau} \frac{\delta}{2}] = \frac{\delta}{2} E[\tau \wedge n].$$

Let us observe that for a symmetric $D_l$-valued random walk with jumps of size $\delta$, $((P^q)^2 - q\delta^2)_{q=0,1,\dots}$ is a martingale. Therefore, due to the stopping theorem for uniformly integrable martingales, if $P$ is in $D_l$ then $\delta^2 E[\tau]$ is equal to $E[(P^\tau)^2 - P^2]$. Since $P^\tau$ belongs almost surely to $\{0,1\}$ and $E[P^\tau] = P$, we get $E[(P^\tau)^2 - P^2] = P(1-P)$. Due to the monotone convergence theorem, we get

$$\lim_{n\to+\infty} C_n^l = \frac{\delta}{2} E[\lim_{n\to+\infty} \tau \wedge n] = \frac{\delta}{2} E[\tau] = P(1-P)\frac{1}{2\delta} = g^l(P)$$

The convergence of $V_n^l(P)$ to $g^l(P)$ is thus proved for $P \in D_l$. Due to the concavity of $V_n^l$ the convergence will hold clearly for all point in $[0,1]$, and the theorem is proved.

Let us observe that the above described strategy $\sigma^{n,l}$ is in fact not an optimal strategy in the game $G_n^l(P)$. The amount $C_n^l(P)$ it guarantees is symmetrical around $\frac{1}{2}$, $C_n^l(P) = C_n^l(1-P)$ while $V_n^l(P)$ is not (see graphs 1 and 2). We have no explicit expression of the optimal strategies in $G_n^l(P)$, but heuristically, these strategies should be close to $\sigma^{n,l}$, at least for large $n$.

As a corollary of theorem 2.3.1, we have the uniform convergence of $\frac{V_n^l}{\sqrt{n}}$ to 0. This indicates that the continuous and the discrete models are quite different. In particular, we do not expect to have the appearance of a Brownian motion as $n$ goes to infinity for a fixed $l$ in the asymptotic dynamics of the price process in the discretized games. More precisely, let us consider player 1's price process in $G_n^l(P)$ when using $\sigma^{n,l}$. Up to an error $\delta$, this process is equal to the a posteriori martingale. As in [4] (see theorem 2.4 in this paper), this a posteriori martingale may be represented by the continuous time process $\Pi^n$, with $\Pi_t^n := P^q$ if $t \in [\frac{q}{n}, \frac{q+1}{n}[$. Now, if $q \geq \tau$, then $P^q \in \{0,1\}$. Therefore $\Pi_t^n \in \{0,1\}$ whenever $t \geq \tau/n$. We get therefore :

**Theorem 2.3.2** *As $n$ increases to $\infty$, the process $\Pi^n$ converges in law to a splitting martingale $\Pi$ that jumps at time $0$ to $0$ or $1$ and then remains constant.*

However, we prove in the last section of the paper that, in some sense, for moderate $n$, the continuous model remains a good approximation of the discrete one : more precisely, we discretize the optimal strategies in the continuous game, and we show that these discretized strategies guarantee $V_n^l(P) - \epsilon$ in $G_n^l(P)$, with $\epsilon$ proportional to $n\delta$. As a consequence, if $l$ depends on $n$, we get that $\frac{V_n^{l(n)}(P)}{\sqrt{n}}$ converge to the same limit as $\frac{V_n^c(P)}{\sqrt{n}}$ whenever $\sqrt{n}/l(n) \to 0$

The next section is devoted to the lemmas used in the proof of theorem 2.3.1 : we analyze the properties of the recursive operator of the game and we find out its positive fixed point $g^l$.

## 2.4 A positive fixed point for $T$

### 2.4.1 Some properties of $T$

We start this section by proving some easy properties of $T$.

Let us first observe that the value $u(P)$ of the average game with antisymmetric payoff matrix $A(P) := PA^H + (1-P)A^L$ is equal to 0. The optimal strategy for both players is the pure strategy $\lfloor P \rfloor$ defined as follows :

**Definition 2.4.1** *For all $P$ in $[0,1]$ :*
*let $\lfloor P \rfloor = [\frac{P}{\delta}]\delta$ and $\lceil P \rceil = \lfloor P \rfloor + \delta$ ( $[x]$ is the highest integer less or equal to $x$).*

If player 1 uses the pure strategy $\lfloor P \rfloor$, independently of $H, L$ in the definition (2.3.1) of $T(g)(P)$, he plays a non revealing strategy ($P^1 = P$). The first stage payoff in $T(g)(P)$ is just the payoff in the average game which is clearly positive. This leads to the following lemma :

**Lemma 2.4.2** *$T$ is increasing and, for all $g$ : $g \leq T(g)$.*

As a consequence, we have :

**Lemma 2.4.3** *A positive fixed point of $T$ is an upper bound for $V_n^l$.*

Let indeed $g$ be a positive fixed point of $T$ then we have for $n = 0$ : $V_0^l = 0 \leq g$. By induction we get next that, if $V_n^l \leq g$, then $V_{n+1}^l = T(V_n^l) \leq T(g) = g$. □

Unfortunately, the fixed points of $T$ are not easy to find, we will therefore bound $T$ from above by an operator $T^*$ and we will apply the next lemma.

**Lemma 2.4.4** *Let $T^*$ such that $T \leq T^*$.*
*Then a fixed point of $T^*$ is a fixed point of $T$.*

Indeed, $g \leq T(g) \leq T^*(g) = g$.

We will next introduce the operator $T^*$.
The definition (2.3.2) of $T(g)(P)$ contains a minimization over player 2's action $j\delta$. If instead of minimizing, Player 2 plays in that formula $j\delta = \lfloor P \rfloor$, we obtain an operator $T^0$ such that $T(g) \leq T^0(g)$, where

$$T^0(g)(P) := \max_{\{(\sigma(i),P(i)) \ st \ E_\sigma[P(i)]=P\}} [\sum_{i=1}^{l} \sigma(i)[1\!\!1_{i\delta > \lfloor P \rfloor}(P(i) - i\delta) + 1\!\!1_{i\delta < \lfloor P \rfloor}(\lfloor P \rfloor - P(i)) + g(P(i))]$$

In turn, whenever $i\delta > \lfloor P \rfloor$ then $P(i) - i\delta \leq P(i) - \lceil P \rceil$. Therefore

$$T(g) \leq T^0(g) \leq T^1(g)$$

where :

$$T^1(g)(P) = \max_{\{(\sigma(i),P(i)) \ st \ E_\sigma[P(i)]=P\}} [\sum_{i=1}^{l} \sigma(i)[1\!\!1_{i\delta > \lfloor P \rfloor}(P(i) - \lceil P \rceil) + 1\!\!1_{i\delta < \lfloor P \rfloor}(\lfloor P \rfloor - P(i)) + g(P(i))]$$

Finally, $(\sigma, P(.))$ generates a probability distribution on $K \times D_l$. As mentioned above, the max in the definition of $T^1(g)$ is in fact a max over all probability distribution $\Pi$ on $K \times D_l$ such that $\Pi[k = H] = P$. A general procedure to generate such probabilities is as follows : given $\sigma, P(.)$ and a one to one mapping $i$ from $L$ to $L$ where $L = \{0, \ldots, l-1\}$, the lottery $\sigma$ is used to select a virtual action $\tilde{\imath}$, player 1 plays in fact $i(\tilde{\imath})$. The state of nature is chosen according to the lottery $P(\tilde{\imath})$. Therefore we infer that

$$T^1(g) = \max_{\{(\sigma_{\tilde{\imath}},P_{\tilde{\imath}}) \ st \ E_\sigma[P_{\tilde{\imath}}]=p\}} \max_{\{i:permutation \ L \to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[1\!\!1_{i(\tilde{\imath})\delta > \lfloor P \rfloor}(P_{\tilde{\imath}} - \lceil P \rceil) + 1\!\!1_{i(\tilde{\imath})\delta < \lfloor P \rfloor}(\lfloor P \rfloor - P_{\tilde{\imath}}) + g(P_{\tilde{\imath}})]$$

Simply by relaxing hypothesis that $i$ is a permutation, we get a new inequality :

$$T^1(g) \leq T^*(g)$$

where,

$$T^*(g) = \max_{\{(\sigma_{\tilde{\imath}}, P_{\tilde{\imath}})\ st\ E_\sigma[P_{\tilde{\imath}}]=p\}} \max_{\{i:L\to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[1\!\!1_{i(\tilde{\imath})\delta>\lfloor P\rfloor}(P_{\tilde{\imath}}-\lceil P\rceil)+1\!\!1_{i(\tilde{\imath})\delta<\lfloor P\rfloor}(\lfloor P\rfloor-P_{\tilde{\imath}})+g(P_{\tilde{\imath}})]$$

The max over $i : L \to L$ in the last formula can be solved explicitly :
Whenever $P_{\tilde{\imath}} \geq (\frac{\lfloor P\rfloor+\lceil P\rceil}{2})$ ( or equivalently $P_{\tilde{\imath}} - \lceil P\rceil \geq \lfloor P\rfloor - P_{\tilde{\imath}}$), $i(\tilde{\imath})$ must be chosen above $\frac{\lfloor P\rfloor}{\delta}$.
Similarly, if $P_{\tilde{\imath}} < (\frac{\lfloor P\rfloor+\lceil P\rceil}{2})$, then $i(\tilde{\imath}) < \frac{\lfloor P\rfloor}{\delta}$.
We obtain in this way that :

**Lemma 2.4.5**

$$\max_{\{i:L\to L\}} \sum_{\tilde{\imath}=1}^{l} \sigma_{\tilde{\imath}}[1\!\!1_{i(\tilde{\imath})\delta>\lfloor P\rfloor}(P_{\tilde{\imath}} - \lceil P\rceil) + 1\!\!1_{i(\tilde{\imath})\delta<\lfloor P\rfloor}(\lfloor P\rfloor - P_{\tilde{\imath}})] = E_\sigma[F_p(P_{\tilde{\imath}})]$$

*with :*

$$F_P(P_{\tilde{\imath}}) = 1\!\!1_{P_{\tilde{\imath}}\geq(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(P_{\tilde{\imath}} - \lceil P\rceil) + 1\!\!1_{P_{\tilde{\imath}}<(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(\lfloor P\rfloor - P_{\tilde{\imath}}).$$

Note that for all $P$ in $[0, 1]$, $F_P{=}F_{\lfloor P\rfloor}$.

The above result leads us to a new expression of $T^*$ : For all $P$ in $[0, 1]$ :

$$T^*(g)(P) = \max_{\{(\sigma_{\tilde{\imath}}, P_{\tilde{\imath}})\ st\ E_\sigma[P_{\tilde{\imath}}]=p\}} E_\sigma[F_{\lfloor P\rfloor}(P_{\tilde{\imath}}) + g(P_{\tilde{\imath}})]$$

**Definition 2.4.6** *The concavification $cav(f)$ of a function $f$ is the smallest concave function higher than $f$ which is concave.*

With that definition, we obtain that :

$$T^*(g)(P) = cav_{P'}(F_{\lfloor P\rfloor}(P') + g(P'))(p)$$

In particular, the fixed point of $T^*$ are concave.

### 2.4.2 A fixed point of $T^*$

In this section, we seek for a fixed point of $T^*$.
$T^*$ is increasing ($g \leq T(g) \leq T^*(g)$). As a consequence :

**Proposition 2.4.7** *$g$ is a fixed point of $T^*$ if and only if*
*$\forall P \in [0, 1]$, $cav_{P'}(F_{\lfloor P\rfloor}(P') + g(P'))(P) \leq g(P)$.*

We will seek for a fixed point $g$ with the particularity that $g = \min_{d\in D_l} g_d$, where for all $d$, $g_d$ linear on $\mathbb{R}$ and for all $P \in [0,1]$ $g(P) = g_{\lfloor P\rfloor}(P)$. ( this means that $g$ is linear between two successive points of $D_l$ )

To prove that $g$ is a fixed point $T^*$, it is sufficient to verify the condition : for all $P$ in $[0,1]$

$$cav_{P'}(F_{\lfloor P\rfloor}(P') + g(P'))(P) \leq g_{\lfloor P\rfloor}(P)$$

Since $g_d$ is linear for all $d$ in $D_l$, and since the concavification of a negative function is negative, we are led to the following lemma :

**Lemma 2.4.8** *If for all $P$ and $P'$ in $[0,1]$,*

$$F_{\lfloor P\rfloor}(P') + g(P') - g_{\lfloor P\rfloor}(P') \leq 0 \tag{2.4.1}$$

*then $g$ is a fixed point of $T^*$.*

We use the equality $g = \min_d(g_d)$ to simplify (2.4.1). The following lemma leads to an explicit expression of a fixed point of $T^*$.

**Lemma 2.4.9** *If for all $P$ and $P'$ in $[0,1]$,*

$$1\!\!1_{P'\leq\lfloor P\rfloor}(\lfloor P\rfloor - P' + g_{\lfloor P\rfloor-\delta}(P') - g_{\lfloor P\rfloor}(P')) + 1\!\!1_{P'\geq\lceil P\rceil}(P' - \lceil P\rceil + g_{\lceil P\rceil}(P') - g_{\lfloor P\rfloor}(P')) \leq 0 \tag{2.4.2}$$

*then $g = \min_{d\in D_l}(g_d)$ is a fixed point of $T^*$. With the convention $g_{-\delta} := g_0$.*

Indeed, since $g = \min_{d\in D_l}(g_d)$, we get for all $P$ and $P'$ in $[0,1]$ :
$g(P') \leq 1\!\!1_{P'\leq\lfloor P\rfloor} g_{\lfloor P\rfloor-\delta}(P') + 1\!\!1_{\lfloor P\rfloor<P'<\lceil P\rceil} g_{\lfloor P\rfloor}(P') + 1\!\!1_{P'\geq\lceil P\rceil} g_{\lceil P\rceil}(P')$,
therefore for all $P$ and $P'$ in $[0,1]$,

$$\begin{aligned} F_{\lfloor P\rfloor}(P') + g(P') - g_{\lfloor P\rfloor}(P') \quad &\leq \quad 1\!\!1_{P'\geq(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(P' - \lceil P\rceil) + 1\!\!1_{P'<(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(\lfloor P\rfloor - P') + \cdots \\ &\cdots + 1\!\!1_{P'\leq\lfloor P\rfloor}(g_{\lfloor P\rfloor-\delta}(P') - g_{\lfloor P\rfloor}(P')) + \cdots \\ &\cdots + 1\!\!1_{P'\geq\lceil P\rceil}(g_{\lceil P\rceil}(P') - g_{\lfloor P\rfloor}(P')) \end{aligned}$$

Since $1\!\!1_{\lceil P\rceil>P'>(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(P' - \lceil P\rceil) \leq 0$ and $1\!\!1_{\lfloor P\rfloor<P'<(\frac{\lfloor P\rfloor+\lceil P\rceil}{2})}(\lfloor P\rfloor - P') \leq 0$, we infer that $g$ is a fixed point of $T^*$ whenever for all $P$ and $P'$ in $[0,1]$,

$$1\!\!1_{P'\leq\lfloor P\rfloor}(\lfloor P\rfloor - P' + g_{\lfloor P\rfloor-\delta}(P') - g_{\lfloor P\rfloor}(P')) + 1\!\!1_{P'\geq\lceil P\rceil}(P' - \lceil P\rceil + g_{\lceil P\rceil}(P') - g_{\lfloor P\rfloor}(P')) \leq 0$$

□

In particular, if the linear functions $g_d$ satisfy to $g_{\lceil P\rceil}(P') = g_{\lfloor P\rfloor}(P') - P' + \lceil P\rceil$ for all $P$ and $P'$ in $[0,1]$ then the function $g^l = \min_d g_d$ verifies the condition of the previous lemma.

The following set of $g_d$ has all those properties :

$\forall i \in \{0, l-1\}$, $\forall P \in [0,1]$, $g_{i\delta}(P) := (\frac{l}{2} - 1 - i)P + i(i+1)\frac{\delta}{2}$

The resulting function $g^l$ may be computed explicitly : $g_{i\delta}(P)$ is a quadratic convex expression of $i\delta$. It is symmetric around $i\delta = P - \delta/2$. The minimum on $i\delta \in D_l$ is thus reached at the point of $D_l$ that is closest to $P - \delta/2$. This point is clearly $i\delta = \lfloor P \rfloor$, and thus

$$g^l(P) = g_{\lfloor P \rfloor}(P) = (l/2 - 1 - \lfloor P \rfloor/\delta)(P - \lfloor P \rfloor) + \lfloor P \rfloor(1 - \lfloor P \rfloor)\frac{1}{2\delta}$$

This is exactly the function $g^l$ introduced in theorem 2.3.1. It is symmetric around $\frac{1}{2}$ on $[0,1]$).

As a consequence of the previous discussion, we get the following theorem

**Theorem 2.4.10** *$g^l$ is a positive fixed point of $T^*$ and thus of $T$.*

We next compute optimal strategies of player 1 in $T(g^l)(P)$ as well as best replies of player 2 :

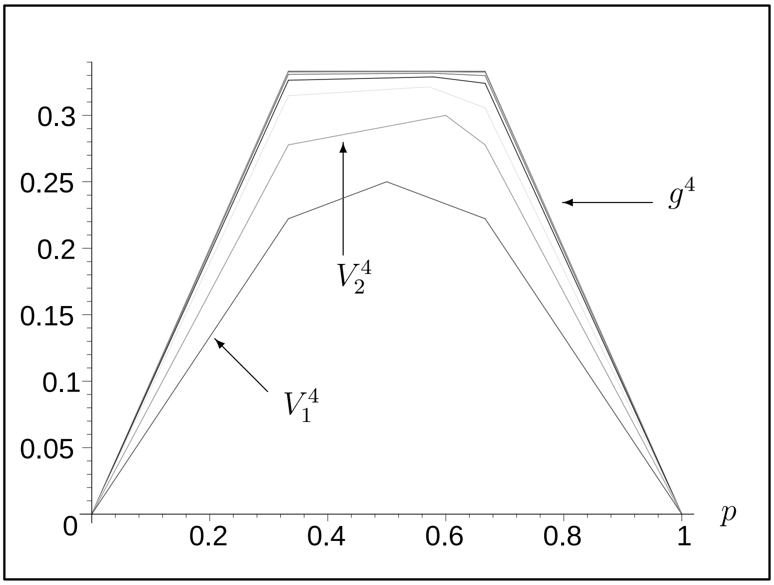
**Theorem 2.4.11** *If $P$ belongs to $D_l \backslash \{0,1\}$ then the following strategy $(\sigma_H, \sigma_L)$ is optimal in $T(g^l)(P)$ : $\sigma_H$ and $\sigma_L$ are lotteries on the prices $P$ and $P^-$ with $\sigma_H(P) = \frac{P^+}{2P}$ and $\sigma_L(P) = \frac{1-P^+}{2(1-P)}$ where $P^+ := P + \delta$ and $P^- := P - \delta$.*
*The best reply of player 2 in $T(g^l)(P)$ against that strategy is to post a price equal to $P$.*

**Proof :**
With that strategy, player 1 plays $P$ with probability $P\sigma_H(P) + (1-P)\sigma_L(P) = \frac{1}{2}$ and therefore $P^1(P)$ is equal to $2P\sigma_H(P) = P^+$. Similarly player 1 plays $P^-$ with probability $P\sigma_H(P^-) + (1-P)\sigma_L(P^-) = \frac{1}{2}$ and therefore $P^1(P^-)$ is equal to $2P\sigma_H(P^-) = P^-$. So, when player 1 uses that strategy, the first stage payoff in $T(g^l)(P)$ is equal to

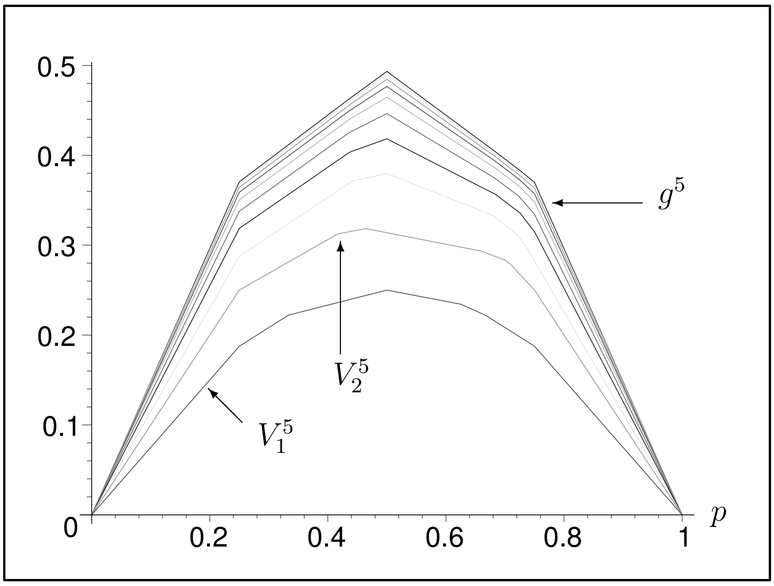
$$\frac{1}{2}[\mathbb{1}_{P>j\delta}(P^+ - P) + \mathbb{1}_{P<j\delta}(j\delta - P^+)] + \frac{1}{2}[\mathbb{1}_{P^->j\delta}(P^- - P^-) + \mathbb{1}_{P^-<j\delta}(j\delta - P^-)]$$

In case $j\delta \leq P^-$, only the first term is not equal to 0 and so the payoff is equal to $\frac{\delta}{2}$. In case $j\delta = P$, only the last term remains and the expectation is also $\frac{\delta}{2}$. The last case to consider is $j\delta \geq P^+$, then we obtain $j\delta - \frac{1}{2}(P^+ + P^-) = j\delta - P \geq \delta$. From this, we obtain that the price $j\delta = P$ is a best reply against that strategy and the first stage payoff is $\frac{\delta}{2}$. The second term payoff is then $\frac{1}{2}g^l(P^+) + \frac{1}{2}g^l(P^-) = g^l(P) - \frac{\delta}{2}$, so as announced the above strategy guarantees $g^l(P) = T(g^l)(P)$ and it is thus optimal in $T(g^l)(P)$.$\square$

**Remark 2.4.12** *The following graphs are drawn from numerical computation of $V_n^l$. It indicates in particular that $V_n^l$ is not symmetric around $\frac{1}{2}$ and thus $V_n^l$ does not coincide with $C_n^l$.*




l=4




l=5

## 2.5 Continuous versus discrete market game

As indicated in the previous section, the continuous and the discrete games are quite different. However, we prove in this section that, in some sense, for moderate $n$, the continuous model remains a good approximation of the discrete one : more precisely, we discretize the optimal strategies in the continuous game, and we show that these discretized strategies guarantee $V_n^l(P) - \epsilon$ in $G_n^l(P)$, with $\epsilon$ proportional to $n\delta$. As a consequence, if $l$ depends on $n$, we get that $\frac{V_n^{l(n)}(P)}{\sqrt{n}}$ converge to the same limit as $\frac{V_n^c(P)}{\sqrt{n}}$ whenever $\sqrt{n}/l(n) \to 0$. This is the content of theorem 2.5.2.
Let us remark that the expression of $V_n^c$ involves the sum of $n$ independent random variables. For $n$ too small ($n < 20$), even in the continuous model, there is not enough independent random variables in these sums for the central limit theorem to be applied. However, as it results from the next theorem, if $l$ is large enough, for middle values of $n$ ($20 < n \ll l$), the continuous game is a good approximation of the discrete game. The discretized optimal strategies of the continuous game are close to be optimal in the discrete game, and the resulting price process will be the discretization of the price process in the continuous game : For $n$ high enough, it involves a Brownian motion.

As reminded in section 2, player 1's strategies in the first stage of $G_n^l$ are represented by a pair $(f_l, Q_l)$ satisfying (1), (2) and (3) of (2.2.1) with the additional requirement on $f_l$ to be $D_l$ valued. We denote $\Gamma_1^l(P)$ the space of these strategies. Similarly player 2 strategy space $\Gamma_2^l$ will be the set of increasing functions $h_l : [0,1] \to D_l$.
In this section we will compare the payoff guaranteed in $G_n^l(P)$ by the discretization $(f_l^\circ, Q_l^\circ)$ (resp $h_l^\circ$) of the optimal strategy $(f^\circ, Q^\circ)$ (resp $h^\circ$ ) in $G_n^c(P)$ to get the next theorem.

**Definition 2.5.1** *If $\lceil x \rceil$ denotes the smallest $d \in D_l$ that dominates $x$, the discretization $\Pi^l(f,Q) := (f_l, Q_l)$ of the strategy $(f,Q)$ is defined as : $f_l := \lceil f \rceil$ and $Q_l(\alpha)$ is the expectation of $Q(u)$ given that $f_l(u) = f_l(\alpha)$ where $u$ is a uniform random variable on $[0,1]$. (Similarly $\Pi^l(h) := \lceil h \rceil$)*

**Theorem 2.5.2** *The discretized optimal strategies of $G_n^c(P)$ are $n\delta$-optimal strategies in $G_n^l(P)$. Therefore :*

$$\forall l, \forall n \geq 1 : \|V_n^c - V_n^l\|_\infty \leq n\delta$$

*where $\delta = \frac{1}{l-1}$.*

With the previous strategy spaces, the recurrence operator $T$ for $V_n^l$, defined in (2.3.2), can be written as :

For all $P \in [0,1]$ :

$$T(g)(P) := sup_{(f,Q)\in\Gamma_1^l(P)} inf_{p_2\in D_l} F((f,Q),p_2,g),$$

with $F$ as in theorem 2.2.1.

**Lemma 2.5.3** *For all $n$ in $\mathbb{N}$, if $(f^\circ, Q^\circ)$ are optimal strategies in the first stage of $G_n^c(P)$, for all $p_2 \in D_l$ :*

$$F((f^\circ,Q^\circ),p_2,V_n^c) \leq F(\Pi^l(f^\circ,Q^*),p_2,V_n^c)+\delta$$

*In particular $T^c(V_n^c) \leq T(V_n^c)+\delta$*

Indeed, if $p_2 \in D_l$ and $(f_l^\circ, Q_l^\circ) := \Pi^l(f^\circ, Q^\circ)$ :

$$\begin{array}{rcl} F((f^\circ,Q^\circ),p_2,V_n^c) & = & E[\mathbb{1}_{f^\circ>p_2}(Q^\circ-f^\circ)+\mathbb{1}_{p_2>f^\circ}(p_2-Q^\circ)+V_n^c(Q^\circ)] \\ & = & E[\mathbb{1}_{f_l^\circ>p_2}(Q^\circ-f^\circ)+\mathbb{1}_{p_2>f_l^\circ}(p_2-Q^\circ)+V_n^c(Q^\circ)] \\ & & +E[\mathbb{1}_{\{f_l^\circ=p_2\&f^\circ<p_2\}}(p_2-Q^\circ)] \\ & = & E[\mathbb{1}_{f_l^\circ>p_2}(Q^\circ-f_l^\circ)+\mathbb{1}_{p_2>f_l^\circ}(p_2-Q^\circ)+V_n^c(Q^\circ)] \\ & & +E[\mathbb{1}_{f_l^\circ>p_2}(f_l^\circ-f^\circ)+\mathbb{1}_{\{f_l^\circ=p_2\&f^\circ<p_2\}}(f_l^\circ-f^\circ)] \\ & & +E[\mathbb{1}_{\{f_l^\circ=p_2\&f^\circ<p_2\}}(f^\circ-Q^\circ)] \end{array}$$

Since we have $0 \leq f_l^\circ - f^\circ \leq \delta$ and as proved on page 298 in [4], $f^\circ - Q^\circ \leq 0$, the second expectation in last equation is clearly bounded by $\delta$ and thus :
$F((f^\circ,Q^\circ),p_2,V_n^c) \leq E[\mathbb{1}_{f_l^\circ>p_2}(Q^\circ-f_l^\circ)+\mathbb{1}_{p_2>f_l^\circ}(p_2-Q^\circ)+V_n^c(Q^\circ)]+\delta$
Since $Q_l^\circ = E[Q^\circ|f_l^\circ]$ and both $\mathbb{1}_{f_l^\circ>p_2}$ and $\mathbb{1}_{f_l^\circ<p_2}$ are $f_l^\circ$ measurable, we may replace $Q^\circ$ by $Q_l^\circ$ in the two first terms of the last inequality. Furthermore, due to Jensen inequality and the concavity of $V_n^c$, we get $E[V_n^c(Q^\circ)] \leq E[V_n^c(E[Q^\circ|f_l^\circ])] = E[V_n^c(Q_l^\circ)]$.
The inequality $F((f^\circ,Q^\circ),p_2,V_n^c) \leq F((f_l^\circ,Q_l^\circ),p_2,V_n^c)+\delta$ follows then immediately.
Finally, since $(f^\circ, Q^\circ)$ is optimal, we have

$$\begin{array}{rcl} T^c(V_n^c) & = & \min_{p_2\in[0,1]} F((f^\circ,Q^\circ),p_2,V_n^c) \\ & \leq & \min_{p_2\in D_l} F((f^\circ,Q^\circ),p_2,V_n^c) \\ & \leq & \min_{p_2\in D_l} F((f_l^\circ,Q_l^\circ),p_2,V_n^c)+\delta \\ & \leq & \max_{(f,Q)\in\Gamma_1^l(P)} \min_{p_2\in D_l} F((f,Q),p_2,V_n^c)+\delta \\ & \leq & T(V_n^c)+\delta \end{array}$$

□

**Proposition 2.5.4** $\forall l$, $\forall n \geq 1$ : $V_n^c - V_n^l \leq n\delta$

The proof is by induction :
The result is clearly true for $n = 0$ ($V_n^c = V_n^l = 0$). Next, if the result is true for $n$ then it holds also for $n + 1$ :
Indeed,

$$\begin{array}{rcl} V_{n+1}^c(P) & = & T^c(V_n^c)(P) \\ & \leq & T(V_n^c)(P) + \delta \\ & \leq & T(V_n^l + n\delta)(P) + \delta \\ & = & T(V_n^l)(P) + (n+1)\delta \\ & = & V_{n+1}^l(P) + (n+1)\delta \end{array}$$

□

To deal with the reverse inequality $\forall l$, $\forall n \geq 1$ : $V_n^l - V_n^c \leq n\delta$, we will work on the dual model :
Let us consider the concave functions $W_n^c$ and $W_n^l$ respectively defined as the Fenchel conjugate of $V_n^c$ and $V_n^l$. Due to (2.2.2), we just have to prove that

$$\forall l, \forall n \geq 1 : W_n^c - W_n^l \leq n\delta$$

These functions are the value of dual games characterized by a recursive structure. The recursive formula for $W_n^c$ was proved in theorem 4.5 in [4], and reminded in theorem 2.2.3. The same argument as in lemma 4.4 in [4], but with $D_l$ valued strategies, gives us a similar recursive formula for $W_n^l$.

$$W_{n+1}^l(x) \geq \Lambda(W_n^l)(x) := sup_{h \in \Gamma_2^l} inf_{p_1 \in D_l} R[x](p_1, h, W_n^l),$$

with $R$ as in theorem 2.2.3.
The inequality in the last formula could be replaced by an equality, and this would lead to the dual recursive formula for the finite games as define in [5].

**Lemma 2.5.5** *For all $x$ in $\mathbb{R}$, for all $n$ in $\mathbb{N}$, if $h^\circ$ is optimal strategy in the first stage of the dual game, for all $p_1 \in D_l$ :*

$$R[x](p_1, h^\circ, W_n^c) - R[x](p_1, \Pi^l(h^\circ), W_n^c) \leq \delta$$

*In particular $\Lambda^c(W_n^c) \leq \Lambda(W_n^c) + \delta$.*

Indeed, with the notation $h_l^\circ := \Pi^l(h^\circ)$ and if $p_1 \in D_l$ :

$$R[x](p_1, h^\circ, W_n^c) = W_n^c(x - \int_0^1 \mathbb{1}_{h^\circ(u) < p_1} - \mathbb{1}_{h^\circ(u) > p_1} du) - \int_0^1 \mathbb{1}_{h^\circ(u) < p_1}(-p_1) + \mathbb{1}_{h^\circ(u) > p_1} h^\circ(u) du$$

To simplify the notations, let us consider $h^\circ(u)$ (with $u$ uniformly distributed) as a random variable $h^\circ$ then $\int_0^1 \mathbb{1}_{h^\circ(u)<p_1} - \mathbb{1}_{h^\circ(u)>p_1} du$ is just equal to $A(h^\circ) :=$

$-1+2Prob(h^\circ < p_1) + Prob(h^\circ = p_1)$.
Next :

$$\begin{array}{rcl} A(h^\circ) & = & -1+2Prob(h_l^\circ < p_1) + 2Prob(h_l^\circ = p_1 \& h^\circ < p_1) + \cdots \\ & & \cdots + Prob(h_l^\circ = p_1) - Prob(h_l^\circ = p_1 \& h^\circ < p_1) \\ & = & -1+2Prob(h_l^\circ < p_1) + Prob(h_l^\circ = p_1) + Prob(h_l^\circ = p_1 \& h^\circ < p_1) \\ & = & A(h_l^\circ) + Prob(h_l^\circ = p_1 \& h^\circ < p_1) \end{array}$$

Therefore, due to the concavity of $W_n^c$ :

$$\begin{array}{rcl} W_n^c(x - A(h^\circ)) & = & W_n^c(x - A(h_l^\circ) - Prob(h_l^\circ = p_1 \& h^\circ < p_1)) \\ & \leq & W_n^c(x - A(h_l^\circ)) - Prob(h_l^\circ = p_1 \& h^\circ < p_1)(W_n^c)'(x - A(h_l^\circ)), \end{array}$$

where $(W_n^c)'$ stands for the derivative of $W_n^c$. Next, $(W_n^c)'(x - A(h_l^\circ)) = (W_n^c)'(x + 1 - 2\zeta)$, with $\zeta := Prob(h_l^\circ < p_1) + \frac{Prob(h_l^\circ = p_1)}{2}$. As proved in formula (18) in [4], $h^\circ(u) = \int_0^u 2s(W_n^c)'(x+1-2s)ds/u^2$.
Due to the concavity of $W_n^c$, $(W_n^c)'$ is a decreasing decreasing function, therefore, if $s \leq u$, then $(W_n^c)'(x+1-2s) \leq (W_n^c)'(x+1-2u)$. We get in this way :

$$h^\circ(u) \leq \int_0^u 2s(W_n^c)'(x+1-2u)ds/u^2 = (W_n^c)'(x+1-2u)$$

and so : $-(W_n^c)'(x - A(h_l^\circ)) \leq -h^\circ(\zeta) \leq -h_l^\circ(\zeta) + \delta$.

We claim next that $Prob(h_l^\circ = p_1 \& h^\circ < p_1) h_l^\circ(\zeta) = Prob(h_l^\circ = p_1 \& h^\circ < p_1)p_1$. Indeed, we just analyze the case $Prob(h_l^\circ = p_1 \& h^\circ < p_1) > 0$ : let us define $x_0 := Prob(h_l^\circ \leq p_1 - \delta)$ and $x_1 := Prob(h_l^\circ \leq p_1)$. Since $h^\circ$ is continuous and increasing and since $x \to \lceil x \rceil$ is left continuous, increasing, $h_l^\circ$ is left continuous, increasing. Therefore $\{u | h_l^\circ(u) \leq p_1 - \delta\}$ is the closed interval $[0, \alpha]$ whose length is precisely $Prob(h_l^\circ \leq p_1 - \delta)$. Therefore $\alpha = x_0$ and thus $h_l^\circ(x_0) \leq p_1 - \delta$. We find similarly $h_l^\circ(x_1) \leq p_1$. Now, since $0 < Prob(h_l^\circ = p_1) = x_1 - x_0$, we infer that on $]x_0, x_1]$, $h_l^\circ$ assumes the constant value $p_1$. Observing that, by definition, $\zeta \in ]x_0, x_1]$, we conclude that $h_l^\circ(\zeta) = p_1$.
Thus,

$$W_n^c(x - A(h^\circ)) \leq W_n^c(x - A(h_l^\circ)) - Prob(h_l^\circ = p_1 \& h^\circ < p_1)(p_1 - \delta)$$

We next deal with the term $-\int_0^1 1\!\!1_{h^\circ(u)<p_1}(-p_1) + 1\!\!1_{h^\circ(u)>p_1} h^\circ(u)du$ in $R(p_1, h^\circ, W_n^c)$. It is just equal to :

$$\begin{array}{l} -\int_0^1 1\!\!1_{h_l^\circ(u)<p_1}(-p_1) + 1\!\!1_{h_l^\circ(u)>p_1} h_l^\circ(u)du + Prob(h_l^\circ = p_1 \& h^\circ < p_1)p_1 + \cdots \\ \cdots + \int_0^1 1\!\!1_{h_l^\circ(u)>p_1}(h_l^\circ(u) - h^\circ(u))du \end{array}$$

Therefore :

$$\begin{array}{rcl} R[x](p_1, h^\circ, W_n^c) & \leq & R[x](p_1, h_l^\circ, W_n^c) + Prob(h_l^\circ = p_1 \& h^\circ < p_1)\delta + \cdots \\ & & \cdots + \int_0^1 1\!\!1_{h_l^\circ(u) > p_1}(h_l^\circ(u) - h^\circ(u))du \end{array}$$

Since $h_l^\circ - h^\circ \leq \delta$, the inequality $R[x](p_1, h^\circ, W_n^c) \leq R[x](p_1, h_l^\circ, W_n^c) + \delta$ follows then immediately.
Finally, since $h^\circ$ is optimal, we have

$$\begin{array}{rcl} \Lambda^c(W_n^c)(x) & = & \min_{p_1 \in [0,1]} R[x](p_1, h^\circ, W_n^c) \\ & \leq & \min_{p_1 \in D_l} R[x](p_1, h^\circ, W_n^c) \\ & \leq & \min_{p_1 \in D_l} R[x](p_1, h_l^\circ, W_n^c) + \delta \\ & \leq & \max_{h \in \Gamma_2^l} \min_{p_1 \in D_l} R[x](p_1, h, W_n^c) + \delta \\ & \leq & \Lambda(W_n^c)(x) + \delta \end{array}$$

□

**Proposition 2.5.6** *$\forall l$, $\forall n \geq 1$ : $W_n^c - W_n^l \leq n\delta$*

The proof is by induction :
The result is clearly true for $n = 0$ ($W_0^c = W_0^l$). If the result is true for $n$ then it holds also for $n + 1$ :
Indeed,

$$\begin{array}{rcl} W_{n+1}^c & = & \Lambda^c(W_n^c) \\ & \leq & \Lambda(W_n^c) + \delta \\ & \leq & \Lambda(W_n^l + n\delta) + \delta \\ & = & \Lambda(W_n^l) + (n+1)\delta \\ & = & W_{n+1}^l + (n+1)\delta \end{array}$$

The result holds thus for all $n$. □

## 2.6 Conclusion

The results of section 3 indicate that the normal density does not appear in the asymptotic behavior of $\Psi_n^l$, as $n$ goes to infinity for a fixed $l$. In particular, we have seen in that case (see theorem 2.3.2) that the limit price process $\Pi$ is a splitting martingale that jumps at time 0 to 0 or 1 and then remains constant. The effect of the discretization is to force the informed player to reveal is information much sooner than in the continuous model. The discretization improves the efficiency of the prices.

Theorem 2.5.2 in terms of $\Psi_n$ reads :

**Corollary 2.6.1** *$\forall l$, $\forall n \geq 0$, $\|\Psi_n^c - \Psi_n^l\|_\infty \leq \frac{\sqrt{n}}{l-1}$*

This implies in particular that if the size $l(n)$ of the discretization set increases with the number $n$ of transaction stages in such a way that $\lim_{n\to+\infty}\frac{l(n)}{\sqrt{n}} = +\infty$, then $\Psi_n^{l(n)}$ converges to the same limit as $\Psi_n^c$, and in that case, the normal distribution does appear. The discretized optimal strategies of the continuous games are then close to be optimal in the discrete game, and the brownian motion will appear in the asymptotic of the price process. Therefore, the continuous game remains a good model for the real world discretized game as far as $\frac{\sqrt{n}}{l-1}$ is small.

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# Chapitre 3

# Repeated games with lack of information on both sides

## 3.1 La théorie des jeux répétés à information incomplète des deux côtés

### 3.1.1 Le modèle

Nous introduisons le modèle de jeux répétés à information incomplète des deux côtés avec des espaces de stratégies finis. Dans les chapitres suivants nous étudierons dans des cas particuliers ce même type de jeux lorsque les joueurs ont des espaces continus d'actions.
Soient $K$ et $L$ des ensembles finis, nous notons $A_k^l$ une famille de matrices de taille $I \times J$, $(k, l)$ dans $K \times L$. La norme de $A$ est définie par $\|A\| := \max_{i,j,k,l} |A_{k,i}^{l,j}|$.
Pour tout $(p, q) \in \Delta(K) \times \Delta(L)$, nous notons $G_n(p, q)$ le jeu suivant :

- A l'étape 0 : la probabilité $p$ (resp. $q$) choisit un état $k$ dans $K$ (resp. $l$ dans $L$), et le joueur 1 (resp. 2) seulement est informé de $k$ (resp. $l$).
- A l'étape $r$, sachant l'histoire passée $h_{r-1} = (i_1, j_1, \ldots, i_{r-1}, j_{r-1})$, les joueurs 1 et 2 choisissent respectivement une action $i_r \in I$ et $j_r \in J$ et la nouvelle histoire $h_r = (i_1, j_1, \ldots, i_r, j_r)$ est annoncée publiquement.

Les joueurs sont informés de la description du jeu. Et nous faisons les notations suivantes :
Nous notons $H_r = (I \times J)^r$ l'ensemble des histoires à l'étape $r$ ($H_0 = \{\emptyset\}$) et $H_n = \cup_{1 \leq r \leq n} H_r$ l'ensemble de toutes les histoires. Nous notons toujours $S = \Delta(I)$ et $T = \Delta(J)$. Une stratégie du joueur 1 (resp. 2) est une application $\sigma$ de $K \times H_n$ dans $S$ (resp. $L \times H_n$ dans $T$). De façon similaire, nous utiliserons la notation $\sigma = (\sigma_1, \ldots, \sigma_n)$ pour le joueur 1 et $\tau = (\tau_1, \ldots, \tau_n)$ pour le joueur 2. Par la suite nous noterons, $\Sigma$ et $\mathcal{T}$ les ensembles de stratégies des joueurs 1 et 2 respective-

ment.

Un élément $(p,q,\sigma,\tau)$ dans $\Delta(K)\times\Delta(L)\times\Sigma\times\mathcal{T}$ induit une probabilité $\Pi_{p,q,\sigma,\tau}$ sur $K\times L\times H_n$ muni de la $\sigma$-algèbre $\mathcal{K}\vee\mathcal{L}\vee_{1\leq r\leq n}\mathcal{H}_r$, où $\mathcal{K}$ (resp. $\mathcal{L}$) est la $\sigma$-algèbre discrète sur $K$ (resp. $L$), et $\mathcal{H}_r$ est la $\sigma$-algèbre naturelle sur l'espace produit $H_r$.
Chaque séquence $(k,l,i_1,j_1,\ldots,i_n,j_n)$ permet d'introduire une suite de paiements $(g_r)_{1\leq r\leq n}$ avec $g_r = A^{l,j_r}_{k,i_r}$. Le paiement du jeu est donc $\gamma_n^{p,q}(\sigma,\tau)=E_{p,q,\sigma,\tau}[\Sigma_{r=1}^n g_r]$. Nous remarquons que le jeu défini est un jeu fini et nous notons $V_n(p,q)$ sa valeur. Nous rappelons que

**Proposition 3.1.1** *$V_n$ est concave en $p$, convexe en $q$ et Lipschitz de rapport $\|A\|$.*

De plus, nous reprenons évidemment les mêmes notions de martingales aposteriori pour le joueur 1 mais également pour le joueur 2. Et nous noterons toujours $V_n^1$ la variation $L^1$.

### 3.1.2 Formule de récurrence

Nous rappelons brièvement le résultat obtenu dans le cadre d'un jeu avec espaces d'actions finis. Nous avons la formule de récurrence suivante pour la valeur $V_n$ :

**Proposition 3.1.2**

$$V_{n+1}(p,q)=\max_{\sigma\in S^K}\min_{\tau\in T^L}\left[\Sigma_{(k,l)\in K\times L}p^k q^l\sigma^k A^l_k\tau^l+\Sigma_{i\in I,j\in J}\bar{\sigma}[i]\bar{\tau}[j]V_n(p_1(i),q_1(j))\right]$$

*avec $\bar{\sigma}=\Sigma_{k\in K}p^k\sigma^k$, $\bar{\tau}=\Sigma_{l\in L}q^l\tau^l$, $p_1(i)=\frac{p^k\sigma^k(i)}{\bar{\sigma}[i]}$ et $q_1(j)=\frac{q^l\tau^l(j)}{\bar{\tau}[j]}$.*

La formule de récurrence est également vraie avec $\min\max$ au lieu de $\max\min$. La formule de récurrence n'apparaît dans la littérature que dans le **cas d'espaces d'actions finis**, et nous remarquons que dans ce cas, la preuve de cette formule n'est pas constructive. En particulier, elle ne nous permet pas d'établir une structure récursive des stratégies optimales des joueurs. La première étape est donc d'exhiber des inégalités de récurrence vérifiées par le maxmin et minmax du jeu répété dans le cadre général, identiques à celles obtenues dans le cadre d'information unilatérale. Nous remarquons également qu'il n'existe pas de formule de récurrence pour les valeurs des jeux duaux (dual du côté du joueur 1 et dual du côté du joueur 2). Ce qui par là même,, ne nous permet pas d'approcher de façon duale les stratégies optimales des joueurs. L'ensemble de ces résultats font l'objet de la section 3.2 intitulée : *"Duality and optimal strategies in the finitely repeated zero-sum games with incomplete information on both sides"*.

### 3.1.3 Comportement asymptotique de $\frac{V_n}{n}$

Notons $u(p,q)$ la valeur du jeu précédent en 1 coup dans lequel aucun des joueurs n'a d'information privée. Dans la suite, nous noterons $\underline{v}_\infty := \liminf_{n\to+\infty} \frac{V_n}{n}$ et $\overline{v}_\infty := \limsup_{n\to+\infty} \frac{V_n}{n}$. Nous remarquons que $\underline{v}_\infty$ et $\overline{v}_\infty$ sont concaves en $p$, convexes en $q$ et Lipschitz de rapport $\|A\|$. Nous avons les résultats suivants :

**Proposition 3.1.3** *Pour tout $p$ dans $\Delta(K)$ et $q$ dans $\Delta(L)$,*

$$\underline{v}_\infty(p,q) \geq cav_p vex_q \left[\max\left\{u(p,q), \underline{v}_\infty(p,q)\right\}\right]$$

$$\overline{v}_\infty(p,q) \leq vex_q cav_p \left[\min\left\{u(p,q), \overline{v}_\infty(p,q)\right\}\right]$$

Nous avons également la propriété variationnelle suivante :

**Proposition 3.1.4** *Soit $f$ une fonction définie sur $\Delta(K) \times \Delta(L)$ vérifiant,*

$$f(p,q) \leq vex_q cav_p \left[\min\left\{u(p,q), f(p,q)\right\}\right]$$

*Alors,*

$$f(p,q) \leq \frac{V_n}{n} + \frac{\|A\|}{n} V_n^1(\mathbf{q})$$

*et donc en particulier, par définition de $\underline{v}_\infty$, $f \leq \underline{v}_\infty$.*

Nous remarquons que $\overline{v}_\infty$ vérifie les hypothèses de la proposition précédente, nous pouvons donc en conclure qu'en appliquant le résultat symétrique pour $\underline{v}_\infty$ que :

**Proposition 3.1.5** *La limite de $\frac{V_n}{n} = v_\infty$ existe et*

$$-\frac{\|A\|}{n} V_n^1(\mathbf{q}) \leq \frac{V_n}{n} - v_\infty \leq \frac{\|A\|}{n} V_n^1(\mathbf{p})$$

Le corollaire immédiat des résultats cités est le suivant : si la valeur $u$ est nulle, alors nous pouvons déduire de la proposition 3.1.3 que

$$0 \leq cav_p vex_q u \leq \underline{v}_\infty \leq \overline{v}_\infty \leq vex_q cav_p u \leq 0$$

Et donc en particulier, $\lim_{n\to+\infty} \frac{V_n}{n} = 0$.

Dans le modèle avec asymétrie bilatérale d'information, il n'existe aucun résultat concernant la convergence de la suite $\frac{V_n}{\sqrt{n}}$. Le chapitre 4 *"Repeated market games with lack of information on both sides"* apporte une réponse à cette question en étudiant la limite de $\frac{V_n}{\sqrt{n}}$ dans le cadre des jeux financiers. Cette limite sera exhibée sous la forme d'un jeu limite semblable à ceux introduis dans *"From repeated games to Brownian games"* (1999) par De Meyer. Cette étude nous permet également de faire apparaître le mouvement Brownien dans le comportement asymptotique de $\frac{V_n}{\sqrt{n}}$ et par là même, d'étendre dans un cas particulier les résultats obtenus dans le cas de manque unilatéral d'information.

## 3.2 Duality and optimal strategies in the finitely repeated zero-sum games with incomplete information on both sides

B. De Meyer and A. Marino

The recursive formula for the value of the zero-sum repeated games with incomplete information on both sides is known for a long time. As it is explained in the paper, the usual proof of this formula is in a sense non constructive : it just claims that the players are unable to guarantee a better payoff than the one prescribed by the formula, but it does not indicates how the players can guarantee this amount.

In this paper we aim to give a constructive approach to this formula using duality techniques. This will allow us to recursively describe the optimal strategies in those games and to apply these results to games with infinite action spaces.

### 3.2.1 Introduction

This paper is devoted to the analysis of the optimal strategies in the repeated zero-sum game with incomplete information on both sides in the independent case. These games were introduced by Aumann, Maschler [1] and Stearns [7]. The model is described as follows : At an initial stage, nature chooses as pair of states $(k,l)$ in $(K \times L)$ with two independent probability distributions $p$, $q$ on $K$ and $L$ respectively. Player 1 is then informed of $k$ but not of $l$ while, on the contrary, player 2 is informed of $l$ but not of $k$. To each pair $(k,l)$ corresponds a matrix $A^l_k := [A^{l,j}_{k,i}]_{i,j}$ in $\mathbb{R}^{I\times J}$, where $I$ and $J$ are the respective action sets of player 1 and 2, and the game $A^l_k$ is the played during $n$ consecutive rounds : at each stage $m = 1,\dots,n$, the players select simultaneously an action in their respective action set : $i_m \in I$ for player 1 and $j_m \in J$ for player 2. The pair $(i_m, j_m)$ is then publicly announced before proceeding to the next stage. At the end of the game, player 2 pays $\sum_{m=1}^{n} A^{l,j_m}_{k,i_m}$ to player 1. The previous description is common knowledge to both players, including the probabilities $p,q$ and the matrices $A^l_k$.

The game thus described is denoted $G_n(p,q)$.

Let us first consider the finite case where $K$, $L$, $I$, and $J$ are finite sets. For a finite set $I$, we denote by $\Delta(I)$ the set of probability distribution on $I$. We also denote by $h_m$ the sequence $(i_1, j_1, \dots, i_m, j_m)$ of moves up to stage $m$ so that $h_m \in H_m := (I \times J)^m$.

A behavior strategy $\sigma$ for player 1 in $G_n(p,q)$ is then a sequence $\sigma = (\sigma_1, \dots, \sigma_n)$

where $\sigma_m : K \times H_{m-1} \to \Delta(I)$. $\sigma_m(k, h_{m-1})$ is the probability distribution used by player 1 to select his action at round $m$, given his previous observations $(k, h_{m-1})$. Similarly, a strategy $\tau$ for player 2 is a sequence $\tau = (\tau_1, \dots, \tau_n)$ where $\tau_m : L \times H_{m-1} \to \Delta(J)$. A pair $(\sigma, \tau)$ of strategies, join to the initial probabilities $(p, q)$ on the sates of nature induces a probability $\Pi^n_{(p,q,\sigma,\tau)}$on $(K \times L \times H_n)$. The payoff of player 1 in this game is then :

$$g_n(p, q, \sigma, \tau) := E_{\Pi^n_{(p,q,\sigma,\tau)}}[\sum_{m=1}^{n} A^{l,j_m}_{k,i_m}],$$

where the expectation is taken with respect to $\Pi^n_{(p,q,\sigma,\tau)}$. We will define $\underline{V}_n(p, q)$ and $\overline{V}_n(p, q)$ as the best amounts guaranteed by player 1 and 2 respectively :

$$\underline{V}_n(p, q) = \sup_\sigma \inf_\tau g_n(p, q, \sigma, \tau) \text{ and } \overline{V}_n(p, q) = \inf_\tau \sup_\sigma g_n(p, q, \sigma, \tau)$$

The functions $\underline{V}_n$ and $\overline{V}_n$ are continuous, concave in $p$ and convex in $q$. They satisfy to $\underline{V}_n(p, q) \le \overline{V}_n(p, q)$. In the finite case, it is well known that, the game $G_n(p, q)$ has a value $V_n(p, q)$ which means that $\underline{V}_n(p, q) = \overline{V}_n(p, q) = V_n(p, q)$. Furthermore both players have optimal behavior strategies $\sigma^*$ and $\tau^*$ :

$$\underline{V}_n(p, q) = \inf_\tau g_n(p, q, \sigma^*, \tau) \text{ and } \overline{V}_n(p, q) = \sup_\sigma g_n(p, q, \sigma, \tau^*)$$

Let us now turn to the recursive structure of $G_n(p, q)$ : a strategy $\sigma = (\sigma_1, \dots, \sigma_{n+1})$ in $G_{n+1}(p, q)$ may be seen as a pair $(\sigma_1, \sigma^+)$ where

$$\sigma^+ = (\sigma_2, \dots, \sigma_{n+1})$$

is in fact a strategy in a game of length $n$ depending on the first moves $(i_1, j_1)$. Similarly, a strategy $\tau$ for player 2 is viewed as $\tau = (\tau_1, \tau^+)$.

Let us now consider the probability $\pi$ (resp. $\lambda$) on $(K \times I)$ (resp. $(L \times J)$) induced by $(p, \sigma_1)$ (resp. $(q, \tau_1)$). Let us denote by $s$ the marginal distribution of $\pi$ on $I$ and let $p^{i_1}$ be the conditional probability on $K$ given $i_1$. Similarly, let $t$ the marginal distribution of $\lambda$ on $J$ and let $q^{j_1}$ be the conditional probability on $L$ given $j_1$.

The payoff $g_{n+1}(p, q, \sigma, \tau)$ may then be computed as follows : the expectation of the first stage payoff is just $g_1(p, q, \sigma_1, \tau_1)$. Conditioned on $i_1, j_1$, the expectation of the $n$ following terms is just $g_n(p^{i_1}, q^{j_1}, \sigma^+(i_1, j_1), \tau^+(i_1, j_1))$. Therefore :

$$g_{n+1}(p, q, \sigma, \tau) = g_1(p, q, \sigma_1, \tau_1) + \sum_{i_1, j_1} s_{i_1} t_{j_1} g_n(p^{i_1}, q^{j_1}, \sigma^+(i_1, j_1), \tau^+(i_1, j_1)). \tag{3.2.1}$$

At a first sight, if $\sigma, \tau$ are optimal in $G_{n+1}(p, q)$, this formula suggests that $\sigma^+(i_1, j_1)$ and $\tau^+(i_1, j_1)$ should be optimal strategies in $G_n(p^{i_1}, q^{j_1})$, leading to the following recursive formula :

**Theorem 3.2.1**
$$V_{n+1} = \underline{T}(V_n) = \overline{T}(V_n)$$
*with the recursive operators $\underline{T}$ and $\overline{T}$ defined as follows :*

$$\underline{T}(f)(p,q) = \sup_{\sigma_1} \inf_{\tau_1} \left\{ g_1(p,q,\sigma_1,\tau_1) + \sum_{i_1,j_1} s_{i_1} t_{j_1} f(p^{i_1}, q^{j_1}) \right\}$$

$$\overline{T}(f)(p,q) = \inf_{\tau_1} \sup_{\sigma_1} \left\{ g_1(p,q,\sigma_1,\tau_1) + \sum_{i_1,j_1} s_{i_1} t_{j_1} f(p^{i_1}, q^{j_1}) \right\}$$

The usual proof of this theorem is as follows : When playing a best reply to a strategy $\sigma$ of player 1 in $G_{n+1}(p,q)$, player 2 is supposed to know the strategy $\sigma_1$. Since he is also aware of his own strategy $\tau_1$, he may compute both a posteriori $p^{i_1}$ and $q^{j_1}$. If he then plays $\tau^+(i_1,j_1)$ a best reply in $G_n(p^{i_1}, q^{j_1})$ against $\sigma^+(i_1,j_1)$, player 1 will get less than $\underline{V}_n(p^{i_1}, q^{j_1})$ in the $n$ last stages of $G_{n+1}(p,q)$. Since player 2 can still minimize the procedure on $\tau_1$, we conclude that the strategy $\sigma$ of player 1 guarantees a payoff less than $\underline{T}(\underline{V}_n)(p,q)$. In other words, $\underline{V}_{n+1} \leq \underline{T}(\underline{V}_n)$. A symmetrical argument leads to $\overline{V}_{n+1} \geq \overline{T}(\overline{V}_n)$.

Next, observe that $\forall f : \overline{T}(f) \geq \underline{T}(f)$. So, using the fact that $G_n$ has a value $V_n$, we get :

$$\overline{V}_{n+1} \geq \overline{T}(\overline{V}_n) = \overline{T}(V_n) \geq \underline{T}(V_n) = \underline{T}(\underline{V}_n) \geq \underline{V}_{n+1}$$

Since $G_{n+1}$ has also a value : $V_{n+1} = \overline{V}_{n+1} = \underline{V}_{n+1}$, the theorem is proved. $\Box$

This proof of the recursive formula is by no way constructive : it just claims that player 1 is unable to guarantee more than $\underline{T}(\underline{V}_n)(p,q)$, but it does not provide a strategy of player 1 that guarantee this amount.

To explain this in other words, the only strategy built in the last proof is a reply $\tau$ of player 2 to a given strategy of player 1. Let us call $\tau^\circ$ this reply of player 2 to an optimal strategy $\sigma^*$ of player 1. $\tau^\circ$ is a best reply of player 2 against $\sigma^*$, but it could fail to be an optimal strategy of player 2. Indeed, it prescribes to play from the second stage on a strategy $\tau^+(i_1,j_1)$ which is an optimal strategy in $G_n(p^{*i_1}, q^{j_1})$, where $p^{*i_1}$ is the conditional probability on $K$ given that player 1 has used $\sigma_1^*$ to select $i_1$. So, if player 1 deviates from $\sigma^*$, the true a posteriori $p^{i_1}$ induced by the deviation may differ from $p^{*i_1}$ and player 2 will still use the strategy $\tau^+(i_1,j_1)$ which could fail to be optimal in $G_n(p^{i_1}, q^{j_1})$. So when playing against $\tau^\circ$, player 1 could have profitable deviations from $\sigma^*$. $\tau^\circ$ would therefore not be an optimal strategy. An example of this kind, where player 2 has no optimal strategy based on the a posteriori $p^{*i_1}$ is presented in exercise 4, in chapter 5 of [5].

An other problem with the previous proof is that it assumes that $G_{n+1}(p,q)$ has a value. This is always the case for finite games. For games with infinite sets of actions however, it is tempting to deduce the existence of the value of $G_{n+1}(p,q)$ from the existence of a value in $G_n$, using the recursive structure. This is the way we proceed in [4]. This would be impossible with the argument in previous proof : we could only deduce that $\overline{V}_{n+1} \geq \overline{T}(V_n) \geq \underline{T}(V_n) \geq \underline{V}_{n+1}$, but we could not conclude to the equality $\overline{V}_{n+1} = \underline{V}_{n+1}$!

Our aim in this paper is to provide optimal strategies in $G_{n+1}(p,q)$. We will prove in theorem 3.2.5 that $\underline{V}_{n+1} \geq \underline{T}(\underline{V}_n)$ by providing a strategy of player 1 that guarantees this amount. Symmetrically, we provide a strategy of player 2 that guarantees him $\overline{T}(\overline{V}_n)$, and so $\overline{T}(\overline{V}_n) \geq \overline{V}_{n+1}$. Since in the finite case, we know by theorem 3.2.1 that $\underline{T}(\underline{V}_n) = V_{n+1} = \overline{T}(\overline{V}_n)$, these strategies are optimal.

These results are also useful for games with infinite action sets : provide one can argue that $\overline{T}(V_n) = \underline{T}(V_n)$, one deduces recursively the existence of the value for $G_{n+1}(p,q)$, since

$$\overline{T}(V_n) = \overline{T}(\overline{V}_n) \geq \overline{V}_{n+1} \geq \underline{V}_{n+1} \geq \underline{T}(\underline{V}_n) = \underline{T}(V_n). \tag{3.2.2}$$

Since our aim is to prepare the last section of the paper where we analyze the infinite action space games, where no general min-max theorem applies to guarantee the existence of $V_n$, we will deal with the finite case as if $\overline{V}_n$ and $\underline{V}_n$ were different functions. Even more, care will be taken in our proofs for the finite case to never use a "min-max" theorem that would not applies in the infinite case.

The dual games were introduced in [2] and [3] for games with incomplete information on one side to describe recursively the optimal strategies of the uninformed player. In games with incomplete information on both sides, both players are partially uninformed. We introduce the corresponding dual games in the next section.

### 3.2.2 The dual games

Let us first consider the amount guaranteed by a strategy $\sigma$ of player 1 in $G_n(p,q)$. With obvious notations, we get :

$$\inf_{\tau} g_n(p,q,\sigma,\tau) = \inf_{\tau=(\tau^1,\dots,\tau^L)} \sum_l q_l \cdot g_n(p,l,\sigma,\tau^l) = \sum_l q_l \cdot y_l(p,\sigma) = \langle q, y(p,\sigma)\rangle,$$

where $\langle\cdot,\cdot\rangle$ stands for the euclidean product in $\mathbb{R}^L$, and

$$y_l(p,\sigma) := \inf_{\tau^l} g_n(p,l,\sigma,\tau^l).$$

The definition of $\underline{V}_n(p,q)$ indicates that

$$\forall p,q : \langle q, y(p,\sigma)\rangle = \inf_{\tau} g_n(p,q,\sigma,\tau) \leq \underline{V}_n(p,q),$$

and the equality $\langle q, y(p,\sigma)\rangle = \underline{V}_n(p,q)$ holds if and only if $\sigma$ is optimal in $G_n(p,q)$. In particular, $\langle q, y(p,\sigma)\rangle$ is then a tangent hyperplane at $q$ of the convex function $q \to \underline{V}_n(p,q)$.

In the following $\partial\underline{V}_n(p,q)$ will denote the under-gradient at $q$ of that function :

$$\partial\underline{V}_n(p,q) := \{y | \forall q' : \underline{V}_n(p,q') \geq \underline{V}_n(p,q) + \langle q' - q, y\rangle\}$$

Our previous discussion indicates that if $\sigma$ is optimal in $G_n(p,q)$, then $y(p,\sigma) \in \partial\underline{V}_n(p,q)$.

As it will appear in the next section, the relevant question to design recursively optimal strategies is as follows : given an affine functional $f(q) = \langle y, q\rangle + \alpha$ such that

$$\forall q : f(q) \leq V_n(p,q), \tag{3.2.3}$$

is there a strategy $\sigma$ such that

$$\forall q : f(q) \leq \langle y(p,\sigma), q\rangle? \tag{3.2.4}$$

To answer this question it is useful to consider the Fenchel transform in $q$ of the convex function $q \to \underline{V}_n(p,q)$ : For $y \in \mathbb{R}^L$, we set :

$$\underline{V}_n^*(p,y) := \sup_q \langle q, y\rangle - \underline{V}_n(p,q)$$

As a supremum of convex functions, the function $\underline{V}_n^*$ is then convex in $(p,y)$ on $\Delta(K) \times \mathbb{R}^L$.

For relation (3.2.3) to hold, one must then have $\alpha \leq -\underline{V}_n^*(p,y)$, so that $\forall q : f(q) \leq \langle y, q\rangle - \underline{V}_n^*(p,y)$.

The function $\underline{V}_n^*(p,y)$ is related the following dual game $G_n^*(p,y)$ : At the initial stage of this game, nature chooses $k$ with the lottery $p$ and informs player 1. Contrary to $G_n(p,q)$, nature does not select $l$, but $l$ is chosen privately by player 2. Then the game proceeds as in $G_n(p,q)$, so that the strategies $\sigma$ for player 1 are the same as in $G_n(p,q)$. For player 2 however, a strategy in $G_n^*(p,y)$ is a pair $(q,\tau)$, with $q \in \Delta(L)$ and $\tau$ a strategy in $G_n(p,q)$. The payoff $g_n^*(p,y,\sigma,(q,\tau))$ paid by player 1 (the minimizer in $G_n^*(p,y)$) to player 2 is then

$$g_n^*(p,y,\sigma,(q,\tau)) := \langle y, q\rangle - g_n(p,q,\sigma,\tau).$$

Let us next define $\underline{W}_n(p,y) = \sup_{q,\tau} \inf_\sigma g_n^*(p,y,\sigma,(q,\tau))$ and $\overline{W}_n(p,y) = \inf_\sigma \sup_{q,\tau} g_n^*(p,y,\sigma,(q,\tau))$.

We then have the following theorem :

**Theorem 3.2.2** $\overline{W}_n(p,y) = \underline{V}_n^*(p,y)$ *and* $\underline{W}_n(p,y) = \overline{V}_n^*(p,y)$.

**Proof:** The following prove is designed to work with infinite action spaces : the "min-max" theorem used here is on vector payoffs instead of on strategies $\sigma$. Let $Y(p)$ be the convex set

$$Y(p) := \{y \in \mathbb{R}^L | \exists\sigma : \forall l : y_l \leq y_l(p,\sigma)\},$$

and let $\overline{Y(p)}$ be its closure in $\mathbb{R}^L$. Then

$$\underline{V}_n(p,q) = \sup_\sigma \langle y(p,\sigma), q\rangle = \sup_{y\in Y(p)} \langle y, q\rangle = \sup_{y\in\overline{Y(p)}} \langle y, q\rangle.$$

Now

$$\overline{W}_n(p,y) = \inf_\sigma \sup_q \left\{\langle y, q\rangle - \inf_\tau g_n(p,q,\sigma,\tau)\right\} = \inf_\sigma \sup_q \langle y - y(p,\sigma), q\rangle$$

Since any $z \in Y(p)$ is dominated by some $y(p,\sigma)$, we find

$$\overline{W}_n(p,y) = \inf_{z\in Y(p)} \sup_q \langle y - z, q\rangle = \inf_{z\in\overline{Y(p)}} \sup_q \langle y - z, q\rangle$$

Next, we may apply the "min-max" theorem for a bilinear functional with two closed convex strategy strategy spaces, one of which is compact, and we get thus

$$\overline{W}_n(p,y) = \sup_q \inf_{z\in\overline{Y(p)}} \langle y - z, q\rangle = \sup_q \{\langle y, q\rangle - \underline{V}_n(p,q)\} = \underline{V}_n^*(p,y)$$

On the other hand,

$$\begin{aligned}
\underline{W}_n(p,y) &= \sup\nolimits_{q,\tau} \inf\nolimits_\sigma \{\langle y, q\rangle - g_n(p,q,\sigma,\tau)\} \\
&= \sup\nolimits_q \{\langle y, q\rangle - \inf\nolimits_\tau \sup\nolimits_\sigma g_n(p,q,\sigma,\tau)\} \\
&= \overline{V}_n^*(p,y)
\end{aligned}$$

This concludes the proof.□

We are now able to answer our previous question : Let $\sigma$ be an optimal strategy of player 1 in $G_n^*(p,y)$. Then, $\forall q,\tau : \overline{W}_n(p,y) \geq \langle y, q\rangle - g_n(p,q,\sigma,\tau)$, therefore, $\forall q$ :

$$\langle y(p,\sigma), q\rangle = \inf_\tau g_n(p,q,\sigma,\tau) \geq \langle y, q\rangle - \underline{V}_n^*(p,y) \geq f(q). \quad (3.2.5)$$

Let us finally remark that if, for some $q$, $y \in \partial\underline{V}_n(p,q)$, then Fenchel lemma indicates that $\underline{V}_n(p,q) = \langle y, q\rangle - \underline{V}_n^*(p,y)$, and the above inequality indicates that $\sigma$ guarantees $\underline{V}_n(p,q)$ in $G_n(p,q)$ :

**Theorem 3.2.3** *Let* $y \in \partial\underline{V}_n(p,q)$*, and let* $\sigma$ *be an optimal strategy of player 1 in* $G_n^*(p,y)$*. Then* $\sigma$ *is optimal in* $G_n(p,q)$*.*

This last result indicates how to get optimal strategies in the primal game, having optimal strategies in the dual one.

### 3.2.3 The primal recursive formula

Let us come back on formula (3.2.1). Suppose $\sigma_1$ is already fixed. Given an array $y_{i,j}$ of vectors in $\mathbb{R}^L$, player 1 may decide to play $\sigma^+(i_1, j_1)$ an optimal strategy in $G_n^*(p^{i_1}, y_{i_1,j_1})$. As indicates relation (3.2.5), for all strategy $\tau^+$ :

$$\begin{array}{rcl} g_n(p^{i_1}, q^{j_1}, \sigma^+(i_1,j_1), \tau^+(i_1,j_1)) & \geq & \langle y(p^{i_1}, \sigma^+(i_1,j_1)), q^{j_1}\rangle \\ & \geq & \langle y_{i_1,j_1}, q^{j_1}\rangle - \underline{V}_n^*(p^{i_1}, y_{i_1,j_1}) \end{array}$$

and so, if $\overline{y}_j := \sum_i s_i y_{i,j}$, formula (3.2.1) gives :

$$g_{n+1}(p,q,\sigma,\tau) \geq g_1(p,q,\sigma_1,\tau_1) + \sum_{j_1} t_{j_1}\langle \overline{y}_{j_1}, q^{j_1}\rangle - \sum_{j_1} t_{j_1}\sum_{i_1} s_{i_1}\underline{V}_n^*(p^{i_1}, y_{i_1,j_1})$$

We now have to indicate how player 1 will chose the array $y_{i,j}$. He will proceed in two steps : suppose $\overline{y}_j$ is fixed, he has then advantage to pick the $y_{i,j}$ among the solutions of the following minimization problem $\Psi(p, \sigma_1, \overline{y}_j)$, where

$$\Psi(p,\sigma_1,\overline{y}) := \inf_{y_i : \overline{y} := \sum_i s_i y_i} \sum_i s_i \underline{V}_n^*(p^i, y_i)$$

**Lemma 3.2.4** *Let $f_{p,\sigma_1}$ be defined as the convex function*

$$f_{p,\sigma_1}(q) := \sum_i s_i \underline{V}_n(p^i, q).$$

*Then the problem $\Psi(p,\sigma_1,\overline{y})$ has optimal solutions and*

$$\Psi(p,\sigma_1,\overline{y}) = f_{p,\sigma_1}^*(\overline{y}). \tag{3.2.6}$$

**Proof:** First of all observe that $\forall q$ : $\underline{V}_n^*(p^i, y_i) \geq \langle y_i, q\rangle - \underline{V}_n(p^i, q)$, and thus $\Psi(p,\sigma_1,\overline{y}) \geq \langle \overline{y}, q\rangle - f_{p,\sigma_1}(q)$. This holds for all $q$, so $\Psi(p,\sigma_1,\overline{y}) \geq f_{p,\sigma_1}^*(\overline{y})$.

On the other hand, let $q^*$ be a solution of the maximization problem :

$$\sup_q \langle \overline{y}, q\rangle - f_{p,\sigma_1}(q),$$

then $\overline{y} \in \partial f_{p,\sigma_1}(q^*)$. Now, the functions $q \to \underline{V}_n(p^i, q)$ are finite on $\Delta(L)$, and we conclude with Theorem 23.8 in [6] that

$$\partial f_{p,\sigma_1}(q^*) = \sum_i s_i \partial \underline{V}_n(p^i, q^*). \tag{3.2.7}$$

In particular, there exists $y_i^* \in \partial \underline{V}_n(p^i, q^*)$ such that $\overline{y} = \sum_i s_i y_i^*$. Now observe that :

$$\begin{array}{rcl} \Psi(p,\sigma_1,\overline{y}) & \leq & \sum_i s_i \underline{V}_n^*(p^i, y_i^*) \\ & = & \sum_i s_i \{\langle y_i^*, q^*\rangle - \underline{V}_n(p^i, q^*)\} \\ & = & \langle \overline{y}, q^*\rangle - f_{p,\sigma_1}(q^*) \\ & = & f_{p,\sigma_1}^*(\overline{y}) \end{array}$$

So both formula (3.2.6) and the optimality of $y_i^*$ are proven. □

Suppose thus that player one picks optimal $y_{i,j}$ in the problem $\Psi(p,\sigma_1,\overline{y}_j)$. He guarantees then :

$$g_{n+1}(p,q,\sigma,\tau) \geq g_1(p,q,\sigma_1,\tau_1) + \sum_{j_1} t_{j_1}\langle \overline{y}_{j_1}, q^{j_1}\rangle - \sum_{j_1} t_{j_1} f^*_{p,\sigma_1}(\overline{y}_{j_1})$$

Next let $A^j_{p,\sigma_1}$ denote the $L$-dimensional vector with $l$-th component equal to

$$A^j_{p,\sigma_1} := \sum_{k,i} p_k \sigma_{1,k,i} A^{l,j}_{k,i}.$$

With this definition, we get $g_1(p,q,\sigma_1,\tau_1) = \sum_{j_1} t_{j_1}\langle A^{j_1}_{p,\sigma_1}, q^{j_1}\rangle$. Therefore :

$$g_{n+1}(p,q,\sigma,\tau) \geq \sum_{j_1} t_{j_1}\langle A^{j_1}_{p,\sigma_1} + \overline{y}_{j_1}, q^{j_1}\rangle - \sum_{j_1} t_{j_1} f^*_{p,\sigma_1}(\overline{y}_j)$$

Suppose next that player 1 picks $y \in \mathbb{R}^L$, and plays $\overline{y}_{j_1} := y - A^{j_1}_{p,\sigma_1}$. Since $\sum_j t_j q^j = q$, the first sum in the last relation will then be independent of the strategy $\tau_1$ of player 2. It follows :

$$\begin{array}{rcl} g_{n+1}(p,q,\sigma,\tau) & \geq & \langle y,q\rangle - \sum_{j_1} t_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) \\ & \geq & \langle y,q\rangle - \sup_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) \end{array} \tag{3.2.8}$$

We will next prove that choosing appropriate $\sigma_1$ and $y$, player 1 can guarantee $\underline{T}(\underline{V}_n)(p,q)$ :

$$\begin{array}{rcl} g_{n+1}(p,q,\sigma,\tau) & \geq & \langle y,q\rangle - \sup_{t\in\Delta(J)} \sum_{j_1} t_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) \\ & = & \langle y,q\rangle - \sup\limits_{\substack{t\in\Delta(J)\\ r^1\ldots r^J\in\Delta(L)}} \sum_{j_1} t_{j_1}\left\{\langle y - A^{j_1}_{p,\sigma_1}, r^{j_1}\rangle - f_{p,\sigma_1}(r^{j_1})\right\} \end{array}$$

Let $\overline{r}$ denote $\sum_{j_1} t_{j_1} r^{j_1}$. The maximization over $t, r$ can be split in a maximization over $\overline{r} \in \Delta(L)$ and then a maximization over $t, r$ with the constraint $\overline{r} = \sum_{j_1} t_{j_1} r^{j_1}$. This last maximization is clearly equivalent to a maximization over a strategy $\tau_1$ of player 2 in $G_1(p,\overline{r})$, inducing a probability $\lambda$ on $(J\times L)$, whose marginal on $J$ is $t$ and the conditional on $L$ are the $r^{j_1}$. In this way, $\sum_{j_1} t_{j_1}\langle A^{j_1}_{p,\sigma_1}, r^{j_1}\rangle = g_1(p,\overline{r},\sigma_1,\tau_1)$, and we get :

$$g_{n+1}(p,q,\sigma,\tau) \geq \inf_{\overline{r}}\left\{\langle y, q-\overline{r}\rangle + H(p,\sigma_1,\overline{r})\right\}$$

where $H(p,\sigma_1,\overline{r}) := \inf_{\tau_1}\left(g_1(p,\overline{r},\sigma_1,\tau_1) + \sum_{j_1} t_{j_1} f_{p,\sigma_1}(r^{j_1})\right)$. We will prove in lemma 3.2.7 that $H(p,\sigma_1,\overline{r})$ is a convex function of $\overline{r}$. If player 1 chooses $y \in \partial H(p,\sigma_1,q)$ then $\forall \overline{r} : \langle y, q-\overline{r}\rangle + H(p,\sigma_1,\overline{r}) \geq H(p,\sigma_1,q)$, and thus

$$g_{n+1}(p,q,\sigma,\tau) \geq H(p,\sigma_1,q)$$

Replacing now $f_{p,\sigma_1}$ by its value, we get :

$$H(p, \sigma_1, q) = \inf_{\tau_1} \left( g_1(p, q, \sigma_1, \tau_1) + \sum_{i_1, j_1} s_{i_1} t_{j_1} \underline{V}_n(p^{i_1}, q^{j_1}) \right) \tag{3.2.9}$$

Since player 1 can still maximize over $\sigma_1$, we just have proved that player 1 can guarantee

$$\sup_{\sigma_1} H(p, \sigma_1, q) \tag{3.2.10}$$

proceeding as follows :

1. He first selects an optimal $\sigma_1$ in (3.2.10), that is, an optimal strategy in the problem $\underline{T}(\underline{V}_n)(p, q)$.
2. He then computes the function $r \to H(p, \sigma_1, r)$ and picks $y \in \partial H(p, \sigma_1, q)$.
3. He next defines $\overline{y}_j$ as $\overline{y}_j = y - A^j_{p,\sigma_1}$ and finds optimal $y_{i,j}$ in the problem $\Psi(p, \sigma_1, \overline{y}_j)$ as in the proof of lemma 3.2.4.
4. Finally, he selects $\sigma^+(i, j)$ an optimal strategy in $G^*_n(p^i, y_{i,j})$.

The next theorem is thus proved.

**Theorem 3.2.5** *With the above described strategy, player 1 guarantees $\underline{T}(\underline{V}_n)(p, q)$ in $G_{n+1}(p, q)$. Therefore : $\underline{V}_{n+1}(p, q) \geq \underline{T}(\underline{V}_n)(p, q)$*

The first part of the proof of theorem 3.2.1 indicates that $\underline{V}_{n+1}(p, q) \leq \underline{T}(\underline{V}_n)(p, q)$, and this result will hold even for games with infinite action spaces : it uses no min-max argument. We may then conclude :

**Corollary 3.2.6** *$\underline{V}_{n+1}(p, q) = \underline{T}(\underline{V}_n)(p, q)$ and the above described strategy is thus optimal in $G_{n+1}(p, q)$.*

It just remains for us to prove the following lemma :

**Lemma 3.2.7** *The function $H(p, \sigma_1, \overline{r})$ is convex in $\overline{r}$.*

**Proof:** Let us denote $\Delta_{\overline{r}}$ the set of probabilities $\lambda$ on $(J \times L)$, whose marginal $\lambda_{|L}$ on $L$ is $\overline{r}$. As mentioned above, a strategy $\tau_1$, joint to $\overline{r}$, induces a probability $\lambda$ in $\Delta_{\overline{r}}$, and conversely, any such $\lambda$ is induced by some $\tau_1$.

Let next $e_l$ be the $l$-th element of the canonical basis of $\mathbb{R}^L$. The mapping $e : l \to e_l$ is then a random vector on $(J \times L)$, and $r^{j_1} = E_\lambda[e|j_1]$. Similarly, the mapping $A_{p,\sigma_1} : (l, j_1) \to A^{l,j_1}_{p,\sigma_1}$ is a random variable and $E_\lambda[A_{p,\sigma_1}] = g_1(p, \overline{r}, \sigma_1, \tau_1)$. We get therefore

$$H(p, \sigma_1, \overline{r}) := \inf_{\lambda \in \Delta_{\overline{r}}} E_\lambda[A_{p,\sigma_1} + f_{p,\sigma_1}(E_\lambda[e|j_1])].$$

Let now $\pi_0, \pi_1 \geq 0$, with $\pi_0 + \pi_1 = 1$, let $\overline{r}_0, \overline{r}_1, \overline{r}_\pi \in \Delta(L)$, with $\overline{r}_\pi = \pi_1 \overline{r}_1 + \pi_0 \overline{r}_0$. Let $\lambda_u \in \Delta_{\overline{r}_u}$, for $u$ in $\{0,1\}$. Then $\pi, \lambda_1, \lambda_0$ induce a probability $\mu$ on $(\{0,1\} \times J \times L)$ : first pick $u$ at random in $\{0,1\}$, with probability $\pi_1$ of $u$ being 1. Then, conditionally to $u$, use the lottery $\lambda_u$ to select $(j_1, l)$. The marginal $\lambda_\pi$ of $\mu$ on $(J \times L)$ is obviously in $\Delta_{\overline{r}_\pi}$. Next observe that, due to Jensen's inequality and the convexity of $f_{p,\sigma_1}$ :

$$\begin{array}{rcl}\sum_u \pi_u E_{\lambda_u}[A_{p,\sigma_1} + f_{p,\sigma_1}(E_{\lambda_u}[e|j_1])] & = & E_\mu[A_{p,\sigma_1} + f_{p,\sigma_1}(E_{\lambda_u}[e|j_1])] \\ & = & E_\mu[A_{p,\sigma_1} + f_{p,\sigma_1}(E_\mu[e|j_1,u])] \\ & \geq & E_\mu[A_{p,\sigma_1} + f_{p,\sigma_1}(E_\mu[e|j_1])] \\ & = & E_{\lambda_\pi}[A_{p,\sigma_1} + f_{p,\sigma_1}(E_{\lambda_\pi}[e|j_1])] \\ & \geq & H(p,\sigma_1,\overline{r}_\pi)\end{array}$$

Minimizing the left hand side in $\lambda_0$ and $\lambda_1$, we obtain :

$$\sum_u \pi_u H(p,\sigma_1,\overline{r}_u) \geq H(p,\sigma_1,\overline{r}_\pi)$$

and the convexity is thus proved. $\square$

### 3.2.4 The dual recursive structure

The construction of the optimal strategy in $G_{n+1}(p,q)$ of last section is not completely satisfactory : the procedure ends up in point 4) by selecting optimal strategies in the dual game $G^*_n(p, y_{i,j})$ but it does not explain how to construct such strategies. The purpose of this section is to construct recursively optimal strategies in the dual game. It turns out that this construction will be "self-contained" and truly recursive : finding optimal strategies in $G^*_{n+1}$ will end up in finding optimal strategies in $G^*_n$.

Given $\sigma_1$, let us consider the following strategy $\sigma = (\sigma_1, \sigma^+)$ in $G^*_{n+1}(p,y)$ : player 1 sets $\overline{y}_j = y - A^j_{p,\sigma_1}$ and finds optimal $y_{i,j}$ in the problem $\Psi(p,\sigma_1,\overline{y}_j)$ as in the proof of lemma 3.2.4. He then plays $\sigma^+(i_1,j_1)$ an optimal strategy in $G^*_n(p, y_{i_1,j_1})$. This is exactly what we prescribed for player 1 in the beginning of last section. In particular, this strategy was not depending on $q$ in the last section, so that inequality (3.2.8) holds for all $q$, $\tau$ :

$$\sup_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) \geq \langle y, q\rangle - g_{n+1}(p,q,\sigma,\tau) = g^*_{n+1}(p,y,\sigma,(q,\tau))$$

So, with lemma 3.2.4, and the definition of $\Psi$.

$$\begin{array}{rcl}g^*_{n+1}(p,y,\sigma,(q,\tau)) & \leq & \sup_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) \\ & = & \sup_{j_1} \Psi(p,\sigma_1, y - A^{j_1}_{p,\sigma_1}) \\ & = & \sup\limits_{j_1} \inf\limits_{y_i:\sum_i s_i y_i = y - A^{j_1}_{p,\sigma_1}} \sum_i s_i \underline{V}^*_n(p^i, y_i) \\ & = & \inf\limits_{y_{i,j}:\sum_i s_i y_{i,j} = y - A^j_{p,\sigma_1}} \sup\limits_{j_1} \sum_i s_i \underline{V}^*_n(p^i, y_{i,j_1})\end{array} \qquad (3.2.11)$$

Notice that there is no "min-max" theorem needed to derive the last equation : We just allowed the variables $y_i$ to depend on $j_1$ : the new variables are $y_{i,j}$.

With theorem 3.2.2, $\underline{V}_n^*(p^i, y_{i,j_1}) = \overline{W}_n(p^i, y_{i,j_1})$. It is next convenient to define $m_{i,j} := y_{i,j} - y + A^j_{p,\sigma_1}$, so that $\sum_i s_i m_{i,j} = 0$, and to take $m_{i,j}$ as minimization variables :

$$g^*_{n+1}(p, y, \sigma, (q, \tau)) \leq \inf_{m_{i,j}:\sum_i s_i m_{i,j}=0} \sup_{j_1} \sum\nolimits_{i_1} s_{i_1} \overline{W}_n(p^{i_1}, y - A^{j_1}_{p,\sigma_1} + m_{i_1,j_1}) \quad (3.2.12)$$

Let still player 1 minimize this procedure over $\sigma_1$. It follows :

**Theorem 3.2.8** *The above defined strategy $\sigma$ guarantees $\overline{T}^*(\overline{W}_n)(p, y)$ to player 1 in $G^*_{n+1}(p, y)$, where, for a convex function $W$ on $(\Delta(K) \times \mathbb{R}^L)$ :*

$$\overline{T}^*(W)(p, y) := \inf_{\substack{\sigma_1 \\ m_{i,j}:\sum_i s_i m_{i,j}=0}} \sup_{j_1} \sum_{i_1} s_{i_1} W(p^{i_1}, y - A^{j_1}_{p,\sigma_1} + m_{i_1,j_1}).$$

*In particular :* $\overline{W}_{n+1}(p, y) \leq \overline{T}^*(\overline{W}_n)(p, y)$

We next will prove the following corollary :

**Corollary 3.2.9** $\overline{W}_{n+1}(p, y) = \overline{T}^*(\overline{W}_n)(p, y)$ *and the strategy $\sigma$ is thus optimal in $G^*_{n+1}(p, y)$.*

**Proof:** If player 1 uses as strategy $\sigma = (\sigma_1, \sigma^+)$ in $G^*_{n+1}(p, y)$, player 2 may reply the following strategy $(q, \tau)$, with $\tau = (\tau_1, \tau^+)$ : for a given choice of $q, \tau_1$, he computes the a posteriori $p^{i_1}, q^{j_1}$ and plays a best reply $\tau^+(i_1, j_1)$ against $\sigma^+(i_1, j_1)$ in $G_n(p^{i_1}, q^{j_1})$. Since

$$g_n(p^{i_1}, q^{j_1}, \sigma^+(i_1, j_1), \tau^+(i_1, j_1)) \leq \underline{V}_n(p^{i_1}, q^{j_1}),$$

we get

$$\begin{aligned} g^*_n(p, y, \sigma, (q, \tau)) &\geq \langle y, q\rangle - g_1(p, q, \sigma_1, \tau_1) - \sum\nolimits_{i_1,j_1} s_{i_1} t_{j_1} \underline{V}_n(p^{i_1}, q^{j_1}) \\ &= \sum\nolimits_{j_1} t_{j_1} \left(\langle y - A^{j_1}_{p,\sigma_1}, q^{j_1}\rangle - \sum\nolimits_{i_1} s_{i_1} \underline{V}_n(p^{i_1}, q^{j_1})\right) \end{aligned}$$

The reply $(q, \tau)$ of player 2 we will consider is that corresponding to the choice of $q, \tau_1$ maximizing this last quantity. This turns out to be a maximization over the joint law $\lambda$ on $(J \times L)$. In turn, it is equivalent to a maximization $(t, q^{j_1})$, without any constraint on $\sum_j t_j q^j$. So :

$$\begin{aligned} g^*_n(p, y, \sigma, (q, \tau)) &\geq \sup\nolimits_t \sum\nolimits_{j_1} t_{j_1} \sup\nolimits_{q^{j_1}} \left(\langle y - A^{j_1}_{p,\sigma_1}, q^{j_1}\rangle - \sum\nolimits_{i_1} s_{i_1} \underline{V}_n(p^{i_1}, q^{j_1})\right) \\ &= \sup\nolimits_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}). \end{aligned}$$

We then derive as in equations (3.2.11) and (3.2.12) that

$$\begin{aligned}\sup_{j_1} f^*_{p,\sigma_1}(y - A^{j_1}_{p,\sigma_1}) &= \inf_{m_{i,j}:\sum_i s_i m_{i,j}=0}\ \sup_{j_1}\ \sum_{i_1} s_{i_1}\overline{W}_n(p^{i_1}, y - A^{j_1}_{p,\sigma_1} + m_{i_1,j_1}) \\ &\geq\ \overline{T}^*(\overline{W}_n)(p,y)\end{aligned}$$

So, player 1 will not be able to guarantee a better payoff in $G^*_{n+1}(y,p)$ than $\overline{T}^*(\overline{W}_n)(p,y)$, and the corollary is proved. □

We thus gave a recursive procedure to construct optimal strategies in the dual game. Now, instead of using the construction of the previous section to play optimally in $G_{n+1}(p,q)$, player 1 can use theorem 3.2.3 : He picks $y \in \partial\underline{V}_{n+1}(p,q)$, and then plays optimally in $G^*_{n+1}(p,y)$, with the recursive procedure introduced in this section.

### 3.2.5 Games with infinite action spaces

In this section, we generalize the previous results to games where $I$ and $J$ are infinite sets. $K$ and $L$ are still finite sets. The sets $I$ and $J$ are then equipped with $\sigma$-algebras $\mathcal{I}$ and $\mathcal{J}$ respectively. We will assume that $\forall k,l$, the mapping $(i,j) \to A^{l,j}_{k,i}$ is bounded and measurable on $(\mathcal{I}\otimes\mathcal{J})$. The natural $\sigma$-algebra on the set of histories $H_m$ is then $\mathcal{H}_m := (\mathcal{I}\otimes\mathcal{J})^{\otimes m}$. A behavior strategy $\sigma$ for player 1 in $G_n(p,q)$ is then a $n$-uple $(\sigma_1,\dots,\sigma_n)$ of transition probabilities $\sigma_m$ from $K\times H_{m-1}$ to $I$ which means : $\sigma_m : (k,h_{m-1},A) \in (K\times H_{m-1}\times\mathcal{I}) \to \sigma_m(k,h_{m-1})[A] \in [0,1]$ satifying $\forall k, h_{m-1} : \sigma_m(k,h_{m-1})[\cdot]$ is a probability measure on $(I,\mathcal{I})$, and $\forall k, A$, $\sigma_m(k,h_{m-1})[A]$ is $\mathcal{H}_m$ measurable. A strategy of player 2 is defined in a similar way. To each $(p,q,\sigma,\tau)$ corresponds a unique probability measure $\Pi^n_{(p,q,\sigma,\tau)}$ on $(K\times L\times H_n, \mathcal{P}(K)\otimes\mathcal{P}(L)\otimes\mathcal{H}_n)$. Since the payoff map $A^{l,j}_{k,i}$ is bounded and measurable, we are allowed to define $g_n(p,q,\sigma,\tau) := E_{\Pi^n_{(p,q,\sigma,\tau)}}[\sum_{m=1}^n A^{l,j_m}_{k,i_m}]$. The definitions of $\overline{V}_n$, $\underline{V}_n$, $\overline{W}_n$ and $\underline{W}_n$ are thus exactly the same as in the finite case, and the a posteriori $p^{i_1}$ and $q^{j_1}$ are defined as the conditional probabilities of $\Pi^1_{(p,q,\sigma_1,\tau_1)}$ on $K$ and $L$ given $i_1$ and $j_1$. The sums in the definition of the recursive operators $\underline{T}$ and $\overline{T}$ are to be replaced by expectations :

$$\underline{T}(f)(p,q) = \sup_{\sigma_1}\inf_{\tau_1}\left\{g_1(p,q,\sigma_1,\tau_1) + E_{\Pi^1_{(p,q,\sigma_1,\tau_1)}}[f(p^{i_1},q^{j_1})]\right\}$$

Let $\mathcal{V}$ denote the set of Lipschitz functions $f(p,q)$ on $\Delta(K)\times\Delta(L)$ that are concave in $p$ and convex in $q$. The result we aim to prove in this section is the next theorem. For all $V\in\mathcal{V}$ such that $\underline{V}_n > V$, we will provide strategies of player 1 that guarantee him $\underline{T}(V)$.

**Theorem 3.2.10** *If $\underline{V}_n \geq V$, where $V\in\mathcal{V}$, then $\underline{V}_{n+1} \geq \underline{T}(V)$.*

**Proof:** Since $\forall \epsilon > 0$, $\underline{T}(V-\epsilon) = \underline{T}(V) - \epsilon$, it is sufficient to prove the result for $V < \underline{V}_n$. In this case, we also have $\forall p, y : V^*(p,y) > \underline{V}_n^*(p,y) = \overline{W}_n(p,y)$.

In the infinite games, optimal strategies may fail to exist. However, due to the strict inequality, $\forall p, y$, there must exist a strategy $\sigma^+_{p,y}$ in $G^*_n(p,y)$ that warrantees strictly less than $V^*(p,y)$ to player 1. Since the payoffs map $A^{l,j}_{k,i}$ is bounded and $V^*$ is continuous, the set $O(p,y)$ of $(p',y') \in \Delta(K) \times \mathbb{R}^L$ such that $\sigma^+_{p,y}$ warrantees $V^*(p',y')$ in $G^*_n(p',y')$ is a neighborhood of $(p,y)$. There exists therefore a sequence $\{(p_m, y_m)\}_{m\in\mathbb{N}}$ such that $\cup_m O(p_m,y_m) = \Delta(K)\times\mathbb{R}^L$. The map $(p,y) \to \sigma^+(p,y)$ defined as $\sigma^+(p,y) := \sigma^+_{p_{m^*},y_{m^*}}$, where $m^*$ is the smallest integer $m$ with $(p,y) \in O(p_m,y_m)$ satisfies then

– for all $\ell$, the map $(p,y) \to \sigma^+_\ell(p,y)(k,h_{\ell-1})$ is a transition probability from $(\Delta(K)\times\mathbb{R}^L\times K\times H_{\ell-1})$ to $I$.
– $\forall p, y : \sigma^+(p,y)$ warrantees $V^*(p,y)$ to player 1 in $G^*_n(p,y)$.

The argument of section 3.2.3 can now be adapted to this setting : Given a first stage strategy $\sigma_1$ and a measurable mapping $y : (i_1,j_1) \to y_{i_1,j_1} \in \mathbb{R}^L$, player 1 may decide to play $\sigma^+(p^{i_1}, y_{i_1,j_1})$ from stage 2 on in $G_{n+1}(p,q)$. Since $\sigma^+(p,y)$ warrantees $V^*(p,y)$ to player 1 in $G^*_n(p,y)$, we get

$$g_n(p^{i_1}, q^{j_1}, \sigma^+(i_1,j_1), \tau^+(i_1,j_1)) \geq \langle y_{i_1,j_1}, q^{j_1}\rangle - V^*(p^{i_1}, y_{i_1,j_1}).$$

Let $s$ and $t$ denote the marginal distribution of $i_1$ and $j_1$ under $\Pi^1_{(p,q,\sigma_1,\tau_1)}$. In the following $E_s[\cdot]$ and $E_t[\cdot]$ are short hand writings for $\int_I \cdot ds(i_1)$ and $\int_J \cdot dt(j_1)$. If $\overline{y}_j := E_s[y_{i,j}]$, formula (3.2.1) gives :

$$g_{n+1}(p,q,\sigma,\tau) \geq g_1(p,q,\sigma_1,\tau_1) + E_t\left[\langle \overline{y}_{j_1}, q^{j_1}\rangle - E_s[V^*(p^{i_1}, y_{i_1,j_1})]\right].$$

As in section 3.2.3, player 1 would have advantage to choose $i_1 \to y_{i_1,j_1}$ optimal in the problem $\Psi(p,\sigma_1,\overline{y}_{j_1})$, where

$$\Psi(p,\sigma_1,\overline{y}) := \inf_{y:\overline{y}:=E_s[y_{i_1}]} E_s[V^*(p^{i_1}, y_{i_1})]$$

Lemma 3.2.4 also holds in this setting, with $f_{p,\sigma_1}(q) := E_s[V(p^{i_1},q)]$. The only difficulty to adapt the prove of section 3.2.3 is to generalize equation (3.2.7). With the Lipschitz property of $V$, we prove in theorem 3.2.12 that there exists a measurable mapping $y : i \to \mathbb{R}^L$ satisfying $E_s[y_{i_1}] = \overline{y}$ and for $s$-a.e $i_1 : y_{i_1} \in \partial V(p^{i_1}, q^*)$. We get in this way $\Psi(p,\sigma_1,\overline{y}) = f^*_{p,\sigma_1}(\overline{y})$.

We next prove that for all measurable map $\overline{y} : j_1 \to \overline{y}_{j_1}$, $\forall \epsilon > 0$, there exists a measurable array $y : (i_1,j_1) \to y_{i_1,j_1}$ such that $\forall j_1 : E_s[y_{i_1,j_1}] = \overline{y}_{j_1}$ and

$$\forall j_1 : E_s[V^*(p^{i_1}, y_{i_1,j_1})] \leq f^*_{p,\sigma_1}(\overline{y}_{j_1}) + \epsilon \tag{3.2.13}$$

The function $f^*_{p,\sigma_1}$ is Lipschitz, and we may therefore consider a triangulation of $\mathbb{R}^L$ in a countable number of $L$-dimensional simplices with small enough diameter to insure that the linear interpolation $\overline{f^*_{p,\sigma_1}}$ of $f^*_{p,\sigma_1}$ at the extreme points of a simplex $S$ satisfies $\overline{f^*_{p,\sigma_1}} \leq f^*_{p,\sigma_1} + \epsilon$ on the interior of $S$. We define then $y(\overline{y}, i)$ on $S \times I$ as the linear interpolation on $S$ of optimal solutions of $\Psi(p, \sigma_1, \overline{y})$ at the extreme points of the simplex $S$. Obviously $E_s[y(\overline{y}, i_1)] = \overline{y}$, and, due to the convexity of $V^*$, we get $E_s[V^*(p^{i_1}, y(\overline{y}, i_1))] \leq \overline{f^*_{p,\sigma_1}}(\overline{y})$. The array $y_{i_1,j_i} := y(\overline{y}_{j_1}, i_1)$ will then satisfy (3.2.13).

With such arrays $y$, Player 1 guarantees up to an arbitrarily small $\epsilon$ :

$$\inf_{\tau_1} g_1(p, q, \sigma_1, \tau_1) + E_t\left[\langle \overline{y}_{j_1}, q^{j_1}\rangle - f^*_{p,\sigma_1}(\overline{y}_{j_1})\right]$$

The proof next follows exactly as in section 3.2.3, replacing summations by expectations.□

As announced in the introduction, the last theorem has a corollary :

**Corollary 3.2.11** *If $\forall V \in \mathcal{V} : \underline{T}(V) = \overline{T}(V) \in \mathcal{V}$, then, $\forall n, p, q$, the game $G_n(p, q)$ has a value $V_n(p, q)$, and $V_{n+1} = \underline{T}(V_n) \in \mathcal{V}$.*

**Proof:** The proof just consists of equation (3.2.2).□

It remains for us to prove the next theorem :

**Theorem 3.2.12** *Let $(\Omega, \mathcal{A}, \mu)$ be probability space, let $U$ be a convex subset of $\mathbb{R}^L$, let $f$ be a function $\Omega \times U \to \mathbb{R}$ satisfying*

– *$\forall \omega$ : the mapping $q \to f(\omega, q)$ is convex.*

– *$\exists M : \forall q, q', \omega : |f(\omega, q) - f(\omega, q')| \leq M|q - q'|$.*

– *$\forall q$ : the mapping $\omega \to f(\omega, q)$ is in $\mathbb{L}^1(\Omega, \mathcal{A}, \mu)$.*

*The function $f_\mu(q) := E_\mu[f(\omega, q)]$ is then clearly convex and $M$-Lipschitz in $q$. Let next $\overline{y} \in \partial f_\mu(q_0)$.*

*Then there exists a measurable map $y : \Omega \to \mathbb{R}^L$ such that*

*1) for $\mu$-a.e. $\omega : y(\omega) \in \partial f(\omega, q_0)$.*

*2) $\overline{y} = E_\mu[y(\omega)]$*

**Proof:** Using a translation, there is no loss of generality to assume $q_0 = 0 \in U$. Then, considering the mapping $g(\omega, q) := f(\omega, q) - f(\omega, 0) - \langle \overline{y}, q\rangle$, and the corresponding $g_\mu(q) := E_\mu[g(\omega, q)]$, we get $\forall \omega : g(\omega, 0) = 0 = g_\mu(0)$ and $\forall q$ : $g_\mu(q) \geq 0$.

Let $\mathcal{S}$ denote the set of $(\alpha, X)$ where $\alpha$ and $X$ are respectively $\mathbb{R}$- and $\mathbb{R}^L$-valued mappings in $\mathbb{L}^1(\Omega, \mathcal{A}, \mu)$. Let us then define

$$\mathcal{R} := \{(\alpha, X) \in \mathcal{S} | E_\mu[\alpha(\omega)] > E_\mu[g(\omega, X(\omega))]\}$$

Our hypotheses on $f$ imply in particular that the map $\omega \to g(\omega, X(\omega))$ is $\mathcal{A}$-measurable and in $\mathbb{L}^1(\Omega, \mathcal{A}, \mu)$. Furthermore the map $X \to E_\mu[g(\omega, X(\omega))]$ is continuous for the $\mathbb{L}^1$-norm, so that $\mathcal{R}$ is an open convex subset of $\mathcal{S}$.

Let us next define the linear space $\mathcal{T}$ as :

$$\mathcal{T} := \{(\alpha, X) \in \mathcal{S} | E_\mu[\alpha(\omega)] = 0, \text{ and } \exists \overline{x} \in \mathbb{R}^L \text{ such that } \mu\text{-a.s. } X(\omega) = \overline{x}\}.$$

Now observe that $\mathcal{R} \cap \mathcal{T} = \emptyset$. Would indeed $(\alpha, X)$ belong to $\mathcal{R} \cap \mathcal{T}$, we would have $\mu$-a.s. $X(\omega) = \overline{x}$, and $0 = E_\mu[\alpha(\omega)] > E_\mu[g(\omega, X(\omega))] = g_\mu(\overline{x}) \geq 0$.

There must therefore exist a linear functional $\phi$ on $\mathcal{S}$ such that

$$\phi(\mathcal{R}) > 0 = \phi(\mathcal{T}).$$

Since the dual of $\mathbb{L}^1$ is $\mathbb{L}^\infty$, there must exist a $\mathbb{R}$-valued $\lambda$ and a $\mathbb{R}^L$-valued $Z$ in $\mathbb{L}^\infty(\Omega, \mathcal{A}, \mu)$ such that

$$\forall (\alpha, X) \in \mathcal{S} : \phi(\alpha, X) = E_\mu[\lambda(\omega)\alpha(\omega) - \langle Z(\omega), X(\omega)\rangle].$$

From $0 = \phi(\mathcal{T})$, it is easy to derive that $E_\mu[Z(\omega)] = 0$ and that $\exists \overline{\lambda} \in \mathbb{R}$ such that $\mu$-a.s. $\lambda(\omega) = \overline{\lambda}$.

Next, $\forall \epsilon > 0$, $\forall X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mu)$, the pair $(\alpha, X)$ belongs to $\mathcal{R}$, where $\alpha(\omega) := g(\omega, X(\omega)) + \epsilon$. So, $\phi(\mathcal{R}) > 0$ with $X \equiv 0$, implies in particular $\overline{\lambda} > 0$, and $\phi$ may be normalized so as to take $\overline{\lambda} = 1$. Finally, we get $\forall \epsilon > 0$, $\forall X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mu) : E_\mu[g(\omega, X(\omega))] + \epsilon > E_\mu[\langle Z(\omega), X(\omega)\rangle]$ and thus, $\forall X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mu) :$ $E_\mu[g(\omega, X(\omega))] \geq E_\mu[\langle Z(\omega), X(\omega)\rangle]$.

For $A \in \mathcal{A}$ and $x \in \mathbb{R}^L$, we may apply the last inequality to $X(\omega) := 1\!\!1_A(\omega)x$, and we get : $E_\mu[1\!\!1_A g(\omega, x)] \geq E_\mu[1\!\!1_A \langle Z(\omega), x\rangle]$. Therefore, for all $x \in \mathbb{R}^L : \mu(\Omega_x) = 1$, where $\Omega_x = \{\omega \in \Omega : g(\omega, x) \geq \langle Z(\omega), x\rangle\}$. So, if $\Omega' := \cap_{x \in \mathbb{Q}^L} \Omega_x$, we get $\mu(\Omega') = 1$, since $\mathbb{Q}^L$ is a countable set, and $\forall \omega \in \Omega'$, $\forall x \in \mathbb{Q}^L : g(\omega, x) \geq \langle Z(\omega), x\rangle$. Due to the continuity of $g(\omega, .)$, the last inequality holds in fact for all $\forall x \in \mathbb{R}^L$, so that $\forall \omega \in \Omega' : Z(\omega) \in \partial g(\omega, 0)$.

Hence, if we define $y(\omega) := \overline{y} + Z(\omega)$, we get $\mu$-a.s. : $y(\omega) \in \partial f(\omega, 0)$ and $E_\mu[y(\omega)] = \overline{y} + E_\mu[Z(\omega)] = \overline{y}$. This concludes the proof of the theorem.□

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# Chapitre 4

# Repeated market games with lack of information on both sides

B. De Meyer and A. Marino

De Meyer and Moussa Saley [8] explains endogenously the appearance of Brownian Motion in finance by modeling the strategic interaction between two asymmetrically informed market makers with a zero-sum repeated game with one-sided information. In this paper, we generalize this model to a setting of a bilateral asymmetry of information. This new model leads us to the analyze of a repeated zero sum game with lack of information on both sides. In De Meyer and Moussa Saley's analysis [8], the appearance of the Brownian motion in the dynamic of the price process is intimately related to the convergence of $\frac{V_n(P)}{\sqrt{n}}$. In the context of bilateral asymmetry of information, there is no explicit formula for the $V_n(p,q)$, however we prove the convergence of $\frac{V_n(p,q)}{\sqrt{n}}$ to the value of a associated "Brownian game", similar to those introduced in [6].

## 4.1 Introduction

Information asymmetries on the financial markets are the subject of an abundant literature in microstructure theory. Initiated by Grossman (1976), Copeland and Galay (1983), Glosten and Milgrom (1985), this literature analyses the interactions between asymmetrically informed traders and market makers. In these very first papers, all the complexity of the strategic use of information is not taken into account : Insiders don't care at each period that their actions reveal information to the uniformed side of the market, they just act in order to maximize their profit at that period, ignoring their profits at the next periods. Kyle (see [13]) is the first to incorporate a strategic use of private information in his model. However, to allow the informed agent to use his information without re-

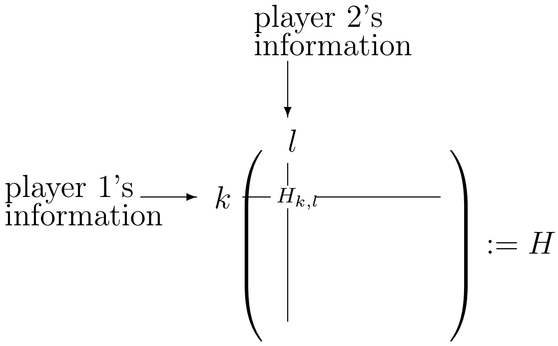
vealing it completely, he introduces noisy traders that play non strategically and that create a noise on insider's actions. A model in which all the agents behave strategically is introduced by De Meyer and Moussa Saley in [8]. In this paper, they consider the interactions between two market markers, one of them is better informed then the other on the liquidation value of the risky asset they trade. In their model, the actions of the agents (the prices they post) are publicly announced, so that the only way for the insider to use his information preserving his informational advantage is to noise his actions. The thesis sustained there is that the sum of these noises introduced strategically to maximize profit will aggregate in a Brownian motion : the one that appears in the price dynamic on the market. All the previous mentioned models only consider the case of one sided information (i.e one agent better informed than the other). In this paper, we aim to generalize De Meyer and Moussa Saley model to a setting of bilateral asymmetry of information.
De Meyer Moussa Saley model turns out to be a zero-sum repeated game with one sided information à la Aumann Maschler but with infinite sets of actions. The main result in Aumann Maschler analysis, the so-called "cav(u)" theorem, identifies the limit of $\frac{V_n}{n}$, where $V_n$ is the value of the $n$-times repeated game. The appearance of the Brownian motion is strongly related to the so-called "error term" analysis in the repeated games literature (see [16], [4], [5] and [6]). These papers analyze for particular games the convergence of $\sqrt{n}\delta_n$, where $\delta_n$ is $\frac{V_n}{n} - cav(u)$. In [8], $cav(u)$ is equal to 0 so that $\sqrt{n}\delta_n = \frac{V_n}{\sqrt{n}}$. De Meyer and Moussa Saley obtain explicit formula for $V_n$ and the convergence of $\frac{V_n}{\sqrt{n}}$ is a simple consequence of the central limit theorem. In this paper, we will have to extend the "error term" for repeated game with incomplete information on both sides. The limit $h$ of $\frac{V_n}{n}$ is identified in [15] as a solution of a system of two functional equations. In this paper, $h$ is equal to 0 and the main result is the proof of the convergence of $\frac{V_n}{\sqrt{n}}$. The proof of this convergence is here much more difficult than in [8] because we don't have explicit formulas for $V_n$. We get this result by introducing a "Brownian game" similar to those introduced in the one side information case in [6].

In [6] and [7], the proof of the convergence of $\sqrt{n}\delta_n$ for a particular class of games is made of three steps : as the first one the value of the Brownian game is proved to exist. The second step is the proof of regularity properties of that value and the fact that it fulfills a partial differential equation, and the last one applies the result of [5] that infers the convergence of $\sqrt{n}\delta_n$ from the existence of a regular solution of the above PDE. In our paper, we proceed differently by proving the global convergence of the $n$-times repeated game to the Brownian game : we don't have to deal with regularity issues nor with PDE.

## 4.2 The model

We consider the interactions between two market makers, player 1 and 2, that are trading two commodities N and R. Commodity N is used as numéraire and has a final value of 1. Commodity R (R for Risky asset) has a final value depending on the state $(k, l)$ of nature $(k, l) \in K \times L$. The final value of commodity R is $H_{k,l}$ in state $(k, l)$,with $H$ a real matrix, by normalization the coefficients of $H$ are supposed to be in $[0, 1]$. By final value of an asset, we mean the conditional expectation of its liquidation price at a fixed horizon T, when $(k, l)$ are made public.

The state of nature $(k, l)$ is initially chosen at random once for all. The independent probability on $K$ and $L$ being respectively $p \in \Delta(K)$ and $q \in \Delta(L)$. Both players are aware of these probabilities. Player 1 (resp. 2) is informed of the resulting state $k$ (resp. $l$) of $p$ (resp. $q$) while player 2 (resp. player 1) is not.




The transactions between the players, up to date $T$, take place during $n$ consecutive rounds. At round $r$ $(r = 1, \dots, n)$, player 1 and 2 propose simultaneously a price $p_{1,r}$ and $p_{2,r}$ in $I = [0, 1]$ for one unit of commodity R. It is indeed quite natural to assume that players will always post prices in $I$ since the final value of $R$ belongs to $I$. The maximal bid wins and one unit of commodity R is transacted at this price. If both bids are equal, no transaction happens. In other words, if $y_r = (y_r^R, y_r^N)$ denotes player 1 's portfolio after round $r$, we have $y_r = y_{r-1} + t(p_{1,r}, p_{2,r})$, with

$$t(p_{1,r}, p_{2,r}) := \mathbb{1}_{p_{1,r}>p_{2,r}}(1, -p_{1,r}) + \mathbb{1}_{p_{1,r}<p_{2,r}}(-1, p_{2,r})$$

The function $\mathbb{1}_{p_{1,r}>p_{2,r}}$ takes the value 1 if $p_{1,r} > p_{2,r}$ and 0 otherwise. At each round the players are supposed to have in memory the previous bids including these of their opponent. The final value of player 1 's portfolio $y_n$ is then $H_{k,l}y_n^R + y_n^N$, and we consider that the players are risk neutral, so that the utility of the players is the expectation of the final value of their own portfolio. Let V denote the final value of player 1's initial portfolio : $V = E[H_{k,l}y_0^R + y_0^N]$. Since $V$ is a constant that does not depend on players' strategies, removing it from player 1's utility function will have no effect on his behavior. This turns out to be equivalent to suppose $y_0 = (0,0)$ ( negative portfolios are then allowed). Similarly, there is no loss of generality to take $(0,0)$ for player 2's initial portfolio . With that convention player 2's final portfolio is just $-y_n$ and player 2's utility is just the opposite of player 1's. We further suppose that both players are aware of the above description. The game thus described will be denoted $G_n(p,q)$. It is essentially a zero-sum repeated game with incomplete information on both sides, just notice that, as compared with Aumann Maschler's model, both players have here at each stage a continuum of possible actions instead of a finite number in the classical model.

## 4.3 The main results of the paper

In this section, we present our main result and explain how the paper is organized. The first result is :

**Theorem 4.3.1** *The game $G_n(p,q)$ has a value $V_n(p,q)$.*
*$V_n(p,q)$ is a concave function of $p \in \Delta(K)$, and a convex function of $q \in \Delta(L)$.*

In the classical model with finite actions sets, the existence of a value and of the optimal strategies for the players was a straightforward consequence of finiteness of the action space. In this framework, this result has to be proved since the players have at each round a continuum of possible actions. More precisely, we will apply the result of [10] on the recursive structure of those games, to get the existence of the value as well as the following recursive formula.

**Theorem 4.3.2** $\forall p \in \Delta(K)$, *and* $\forall q \in \Delta(L)$,

$$V_{n+1}(p,q) = \max_{P \in \mathcal{P}(p)} \min_{Q \in \mathcal{Q}(q)} \int_0^1 \int_0^1 sg(u-v)P(u)HQ(v) + V_n(P(u),Q(v))dudv$$

*with for all $x \in \mathbb{R}$, $sg(x) := 1\!\!1_{x>0} - 1\!\!1_{x<0}$ and*

$$\begin{array}{c} \mathcal{P}(p) := \{P : [0,1] \to \Delta(K) | \int_0^1 P(u)du = p\} \\ \mathcal{Q}(q) := \{Q : [0,1] \to \Delta(L) | \int_0^1 Q(v)dv = q\} \end{array} \tag{4.3.1}$$

Applying this formula recursively, we conclude that $V_n$ is the value of a game in which the players control their a-posteriori martingales, starting respectively from $p$ and $q$ for player 1 and 2. More precisely, we first define the $\sigma$-algebras corresponding essentially to the information available to players at each stage :
Let $(u_1,\ldots,u_n,v_1,\ldots,v_n)$ be a system of independent random variables uniformly

distributed on $[0,1]$ and let us define the filtrations $G^1 := \{G^1_k\}_{k=1}^n$ and $G^2 := \{G^2_k\}_{k=1}^n$ as

$$G^1_k := \sigma(u_1, \ldots, u_k, v_1, \ldots, v_{k-1})$$
$$G^2_k := \sigma(u_1, \ldots, u_{k-1}, v_1, \ldots, v_k)$$

Let also $\mathcal{G} := \{\mathcal{G}_k\}_{k=1}^n$ with $\mathcal{G}_k := \sigma(G^1_k, G^2_k)$. So, the past information available for player $i$ at stage $k$ is then $G^i_k$. In this context, strategies of the game of length $n$ are defined as follow

**Definition 4.3.3**

1. *Let $M^1_n(\mathcal{G}, p)$ the set of $\Delta(K)$-valued $\mathcal{G}$-martingales $X = (X_1, \ldots, X_n)$ that are $G^1$-adapted and satisfying $E[X_1] = p$.*
2. *Similarly, let $M^2_n(\mathcal{G}, q)$ the set of all $\Delta(L)$-valued $\mathcal{G}$-martingales $Y = (Y_1, \ldots, Y_n)$ that are $G^2$-adapted and satisfying $E[Y_1] = q$.*

We thus obtain

**Theorem 4.3.4** *$\forall p \in \Delta(K)$, $\forall q \in \Delta(L)$,*

$$V_n(p,q) = \max_{P \in M^1_n(\mathcal{G},p)} \min_{Q \in M^2_n(\mathcal{G},q)} E[\sum_{i=1}^n sg(u_i - v_i) P_n H Q_n]$$

We now focus our analysis on the asymptotic behavior of the value. The main result of Mertens Zamir (see [15]) relatively to repeated game with lack of information on both sides is the convergence of the value $\frac{V_n}{n}$ to a function $h$ fulfilling the following variational inequalities

$$\begin{cases} cav_p vex_q u \leq h \\ h \leq vex_q cav_p u \end{cases}$$

where $u$ is the value of the 1-round game where no player is informed. In our framework, this game is a symmetric zero-sum game and its value $u$ is thus 0. Hence, $h$ is also equal to 0 in our case.
We are concerned in this paper with a stronger result than the convergence of $\frac{V_n}{n}$ to 0 : we will prove the convergence of $\frac{V_n}{\sqrt{n}}$ to a finite limit $W^c$. To get this result we first introduce the value $W_n$ of a slightly transform game :
For all $p \in \Delta(K)$, $q \in \Delta(L)$

$$W_n(p,q) = \max_{P \in M^1_n(\mathcal{G},p)} \min_{Q \in M^2_n(\mathcal{G},p)} E[\sum_{i=1}^n 2(u_i - v_i) PHQ]$$

The following theorem indicates that the initial game and the modified one are close to each others.

**Theorem 4.3.5** *There exists a constant $C > 0$ such that, for all $n$,*

$$\|V_n - W_n\|_\infty \leq C$$

The advantage of introducing the $W_n$ is that two independent sums of i.i.d random variables : $\sum_{i=1}^n (2u_i - 1)$ and $\sum_{i=1}^n (2v_i - 1)$ appear in the its definition. According to Donsker's theorem, these normalized sums converge in law to two independents Brownian Motions $\beta^1$ and $\beta^2$. Therefore, we get, quite heuristically, the following definition of the continuous "Brownian game".

**Definition 4.3.6** *Let $\mathcal{F}_t^1 := \sigma(\beta_s^1, s \leq t)$ and $\mathcal{F}_t^2 := \sigma(\beta_s^2, s \leq t)$ their natural filtrations and let $\mathcal{F}_t := \sigma(\beta_s^1, \beta_s^2, s \leq t)$. We denote by $\mathcal{H}^2(\mathcal{F})$ the set of $\mathcal{F}_t$-progressively measurable process $a$ such that :*

$$(1) \quad \|a\|_{\mathcal{H}^2}^2 = E[\int_0^{+\infty} a_s^2 ds] < +\infty$$

$$(2) \quad \text{for all } s > 1 : a_s = 0.$$

**Definition 4.3.7 (Brownian game)**
*The Brownian game $G^c(p,q)$ is then defined as the following zero-sum game :*

- *The strategy space of player 1 is the set*

$$\Gamma^1(p) := \left\{ (P_t)_{t\in\mathbb{R}^+} \;\middle|\; \begin{array}{l} \forall t \in \mathbb{R}^+,\ P_t \in \Delta(K), \exists a \in \mathcal{H}^2(\mathcal{F}) \\ \text{such that } P_t := p + \int_0^t a_s d\beta_s^1 \end{array} \right\}$$

- *Similarly, the strategy space of player 2 is the set*

$$\Gamma^2(q) := \left\{ (Q_t)_{t\in\mathbb{R}^+} \;\middle|\; \begin{array}{l} \forall t \in \mathbb{R}^+,\ Q_t \in \Delta(L), \exists b \in \mathcal{H}^2(\mathcal{F}) \\ \text{such that } Q_t := q + \int_0^t b_s d\beta_s^2 \end{array} \right\}$$

- *The payoff function of player 1 corresponding to a pair $P$, $Q$ is*

$$E[(\beta_1^1 - \beta_1^2) P_1 H Q_1]$$

We first prove that the value $W^c(p,q)$ of this continuous game exists. And we then prove that :

**Theorem 4.3.8** *Both sequences $\frac{W_n}{\sqrt{n}}$ and $\frac{V_n}{\sqrt{n}}$ converge uniformly to $W^c$.*

This paper is mainly devoted to the proof of the last convergence result, the analysis of $W^c$ as well as of the optimal martingales, that should in fact be related to the asymptotic behavior of the price system, will be analyzed in a forthcoming paper. So, we don't have a closed formula for $W^c$ except maybe in very particular cases, where the matrix $H$ is of the form $H := x \oplus y := (x_i + y_j)_{i,j}$ with $x \in \mathbb{R}^K$ and $y \in \mathbb{R}^L$. These particular games turn out to be equivalent to playing two separated games with one sided information. Indeed, $P_n H Q_n$ in the formula of $V_n$ becomes $\langle P_n, x\rangle + \langle Q_n, y\rangle$ and so : For all $p \in \Delta(K)$, $q \in \Delta(L)$

$$V_n(p,q) = V_n^x(p) - V_n^y(q)$$

Where $V_n^x$ is the value of repeated market game with one sided information for which $x$ is the final value of $R$. The explicit formula for $V_n$ and the optimal strategies can be found in [8] and [9].

In the next section, we first define the strategy spaces in $G_n(p,q)$, and we next analyze the recursive structure of this game.

## 4.4 The recursive structure of $G_n(p,q)$

### 4.4.1 The strategy spaces in $G_n(p,q)$

Let $h_r$ denote the sequence

$$h_r := (p_{1,1}, p_{2,1}, \ldots, p_{1,r}, p_{2,r})$$

of the proposed prices up to round $r$. When playing round $r$, player 1 has observed $(k, h_{r-1})$. A strategy to select $p_{1,r}$ is thus a probability distribution $\sigma_r$ on $I$ depending on $(k, h_{r-1})$. This leads us to the following definition :

**Definition 4.4.1** A strategy for player 1 in $G_n(p,q)$ is a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_r$ is a transition probability from $(K \times I^{2(r-1)})$ to $(I, \mathcal{B}_I)$ (i.e. a mapping from $(K \times I^{2(r-1)})$ to the set $\Delta(I)$ of probabilities on the Borel $\sigma$-algebra $\mathcal{B}_I$ on $I$, such that $\forall A \in \mathcal{B}_I : \sigma_r(.)[A]$ is measurable on $(K \times I^{2(r-1)})$.)
Similarly, a strategy $\tau$ for player 2 is a sequence $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_r$ is a transition probability from $(L \times I^{2(r-1)})$ to the set to $(I, \mathcal{B}_I)$.

The initial probabilities $p$ and $q$ joint to a pair $(\sigma, \tau)$ of strategies induce inductively a probability distribution $\Pi_n(p,q,\sigma,\tau)$ on $(K \times L \times I^{2n})$. The payoff $g_n(p,q,\sigma,\tau)$ of player 1 corresponding to a pair of strategies $(\sigma,\tau)$ in $G_n(p,q)$ is then :

$$g_n(p,q,\sigma,\tau) = E_{\Pi_n(p,q,\sigma,\tau)}[\langle (H_{k,l}, 1), y_n\rangle].$$

The maximal payoff $V_{1,n}(p,q)$ player 1 can guarantee in $G_n(p,q)$ is

$$V_{1,n}(p,q) := \sup_{\sigma} \inf_{\tau} g_n(p,q,\sigma,\tau).$$

A strategy $\sigma^*$ is optimal for player 1 if $V_{1,n}(p,q) = inf_\tau g_n(p,q,\sigma^*,\tau)$.

Similarly, the better payoff player 2 can guarantee is

$$V_{2,n}(p,q) := \inf_\tau \sup_\sigma g_n(p,q,\sigma,\tau),$$

and an optimal strategy $\tau^*$ for a player 2 is such that $V_{2,n}(p,q) = \sup_\sigma g_n(p,q,\sigma,\tau^*)$. The game $G_n(p,q)$ is said to have a value $V_n(p,q)$ if $V_{1,n}(p,q) = V_{2,n}(p,q) = V_n(p,q)$.

**Proposition 4.4.2** *$V_{1,n}$ and $V_{2,n}$ are concave-convex functions, which means concave in $p$ and convex in $q$. And $V_{1,n} \leq V_{2,n}$.*

The argument is classical for general repeated games with incomplete information and will not be reproduced here (sees [14]).

### 4.4.2 The recursive structure of $G_n(p,q)$.

We are now ready to analyze the recursive structure of $G_n(p,q)$ : after the first stage of $G_{n+1}(p,q)$ has been played, the remaining part of the game is essentially a game of length $n$. Such an observation leads to a recursive formula of the value $V_n$ of the $n$-stages game. At this level of our analysis however we have no argument to prove the existence of $V_n$ and we are only able to provide recursively a lower bound for $V_{1,n+1}(p,q)$. This is the content of theorem 4.4.4. Let us now consider a strategy $\sigma$ of player 1 in $G_{n+1}(p,q)$. The first stage strategy $\sigma_1$ is a conditional probability on $p_{1,1}$ given $k$. Joint to $p$ it induces a probability distribution $\pi_1(p,\sigma_1)$ on $(k,p_{1,1})$ such that : for all $\bar{k}$ in $K$, $\pi_1(p,\sigma_1)[k=\bar{k}] = p^{\bar{k}}$. The remaining part $(\sigma_2,...,\sigma_{n+1})$ of player 1's strategy $\sigma$ in $G_{n+1}(p,q)$ is in fact a strategy $\tilde{\sigma}$ in $G_n$ depending on the first stage actions $(p_{1,1},p_{2,1})$. In the same way, the first stage strategy $\tau_1$ is a conditional probability on $p_{2,1}$ given $l$. Joint to $q$ it induces a probability distribution $\pi_2(q,\tau_1)$ on $(l,p_{2,1})$ such that : : for all $\bar{l}$ in $L$, $\pi_2(q,\tau_1)[l=\bar{l}] = q^{\bar{l}}$.

A strategy $\tau$ of player 2 in $G_{n+1}(p,q)$ can be viewed as a pair $(\tau_1,\tilde{\tau})$, where $\tau_1$ is the first stage strategy, and $\tilde{\tau}$ is a strategy in $G_n$ depending on $(p_{1,1},p_{2,1})$. Let $P(p_{1,1})^{\bar{k}}$ denote $\pi_1(p,\sigma_1)[k=\bar{k}|p_{1,1}]$, and $Q(p_{2,1})^{\bar{l}}$ denote $\pi_2(q,\tau_1)[l=\bar{l}|p_{2,1}]$.

Since $p_{2,1}$ is independent of $k$ and $p_{1,1}$ is independent of $l$, we also have $\Pi_{n+1}(p,q,\sigma,\tau)[k=\bar{k}|p_{1,1},p_{2,1}] = P(p_{1,1})^{\bar{k}}$ and $\Pi_{n+1}(p,q,\sigma,\tau)[l=\bar{l}|p_{1,1},p_{2,1}] = Q(p_{2,1})^{\bar{l}}$. Then, conditionally on $(p_{1,1},p_{2,1})$, the distribution of

$$(k,l,p_{1,2},p_{2,2},\ldots,p_{1,n+1},p_{2,n+1})$$

is $\Pi_n(P(p_{1,1}),Q(p_{2,1}),\tilde{\sigma}(p_{1,1},p_{2,1}),\tilde{\tau}(p_{1,1},p_{2,1}))$.
Therefore $g_{n+1}(p,q,\sigma,\tau)$ is equal to

$$g_1(p,q,\sigma_1,\tau_1)+E_{\Pi_n(p,q,\sigma_1,\tau_1)}[g_n(P(p_{1,1}),Q(p_{2,1}),\tilde{\sigma}(p_{1,1},p_{2,1}),\tilde{\tau}(p_{1,1},p_{2,1})].$$

With that formula in mind, we next define the recursive operators : $\underline{T}$ and $\overline{T}$.

**Definition 4.4.3**

- *Let $\mathcal{M}^{K,L}$ be the space of bounded measurable function $\Psi:\Delta(K)\times\Delta(L)\to\mathbb{R}$.*
- *Let $\mathcal{L}^{K,L}$ be the space of functions $\Psi:\Delta(K)\times\Delta(L)\to\mathbb{R}$ that are Lipschitz on $\Delta(K)\times\Delta(L)$ for the norm $\|.\|$ and concave in $p\in\Delta(K)$, convex in $q\in\Delta(L)$. The norm $\|.\|$ is defined by*

$$\|(p,q)-(\tilde{p},\tilde{q})\|:=\sum_{k\in K}|p^k-\tilde{p}^k|+\sum_{l\in L}|q^l-\tilde{q}^l|.$$

- *Let us then define the functional operators $\overline{T}$ and $\underline{T}$ on $\mathcal{M}^{K,L}$ by :*

$$\underline{T}(\Psi):=\max_{\sigma_1}\min_{\tau_1} g_1(p,q,\sigma_1,\tau_1)+E_{\Pi(p,q,\sigma_1,\tau_1)}[\Psi(P(p_{1,1}),Q(p_{2,1}))] \quad (4.4.1)$$

$$\overline{T}(\Psi):=\min_{\tau_1}\max_{\sigma_1} g_1(p,q,\sigma_1,\tau_1)+E_{\Pi(p,q,\sigma_1,\tau_1)}[\Psi(P(p_{1,1}),Q(p_{2,1}))] \quad (4.4.2)$$

As indicated in theorem 3.2.10 in section 3.2, the above description yields the following recursive inequalities

**Theorem 4.4.4**
*For all $n\in\mathbb{N}$, for all $\Psi\in\mathcal{L}^{K,L}$, $V_{1,n}\geq\Psi\Longrightarrow V_{1,n+1}\geq\underline{T}(\Psi)$.*
*Similarly, for all $n\in\mathbb{N}$, for all $\Psi\in\mathcal{L}^{K,L}$, $V_{2,n}\leq\Psi\Longrightarrow V_{2,n+1}\leq\overline{T}(\Psi)$.*

Notice that, as compared with Aumann-Maschler recursive formula, we only get inequalities at this level. They will proved in corollary 4.4.17 to be equalities.

### 4.4.3 Another parameterization of players' strategy space

In this section, we aim to provide a technically more tractable form for the operators $\overline{T}$ and $\underline{T}$ defined by (4.4.1) and (4.4.2). We will use another parametrization of players strategies.
The first stage strategy space of player 1 may be identified with the space of probability distributions $p$ on $(k,p_{1,1})$ satisfying

$$\pi[k=\bar{k}]=p^{\bar{k}} \quad (4.4.3)$$

In turn, such a probability $\pi$ may be represented as a pair of functions $(f,P)$ : with $f:[0,1]\to[0,1]$ and $P:[0,1]\to\Delta(K)$ satisfying :

$$\begin{array}{ll} a) & f \text{ is increasing} \\ b) & \int_0^1 P(u)du = p \\ c) & \forall x,y\in[0,1] : f(x)=f(y)\Rightarrow P(x)=P(y). \end{array} \tag{4.4.4}$$

Given such a pair $(f,P)$, player 1 generates the probability $\pi$ as follows : he first selects a random number $u$ uniformly distributed on $[0,1]$, he plays then $p_{1,1}:=f(u)$ and he then chooses $k\in K$ at random with a lottery such that $p[k=\bar{k}]=P^{\bar{k}}(u)$.
Notice that any probability $\pi$ satisfying (4.4.3) may be generated in this way. Indeed, if $f$ is the left inverse of the distribution function $F$ of the marginal of $\pi$ on $p_{1,1}$, then $f(u)$ will have the same law as $p_{1,1}$. $f$ is clearly increasing.
Next, let $R(p_{1,1})$ denote $R^{\bar{k}}(p_{1,1}):=\pi[k=\bar{k}|p_{1,1}]$, and let $P(u)$ be defined as $P(u):=R(f(u))$. This pair $(f,P)$ generates $\pi$, and $P$ satisfy clearly to (4.4.4)-c). Finally, (4.4.3) implies (4.4.4)-b). So, we may now view player 1's first stage strategy space as the set of functions $(f,P)$ satisfying (4.4.4).

The question we address now is how to retrieve the first stage strategy $\sigma_1=(\sigma_1(k))_{k\in K}$ from its representation $(f,P)$. If $A\in\mathcal{B}_I$, $\sigma_1(\bar{k})[A]$ is just equal to $\pi[p_{1,1}\in A|k=\bar{k}]=\pi[p_{1,1}\in A\cap k=\bar{k}]/\pi[k=\bar{k}]=\int_0^1 1\!\!1_{f(u)\in A}P^{\bar{k}}(u)du/p^{\bar{k}}$. Therefore, if player 1 is told $\bar{k}$, he picks a random number $u$ in $[0,1]$ according to a probability density $P^{\bar{k}}(u)/p^{\bar{k}}$, and he plays $p_{1,1}=f(u)$.

In the same way, the first stage strategy space of player 2 may be identified with the space of $(g,Q)$ : with $g:[0,1]\to[0,1]$ and $Q:[0,1]\to\Delta(L)$ satisfying :

$$\begin{array}{ll} a) & g \text{ is increasing} \\ b) & \int_0^1 Q(v)dv = Q \\ c) & \forall x,y\in[0,1] : g(x)=g(y)\Rightarrow Q(x)=Q(y). \end{array} \tag{4.4.5}$$

We next proceed to the transformation of the recursive operators (4.4.1) and (4.4.2) :
If player 1 plays the strategy $\sigma_1$ represented by $(f,P)$ and if player 2 plays the strategy $\tau_1$ represented by $(g,Q)$, then $g_1(p,q,\sigma_1,\tau_1)$ is equal to

$$\int_0^1\int_0^1 1\!\!1_{f(u)>g(v)}(P(u)HQ(v)-f(u))+1\!\!1_{f(u)<g(v)}(g(v)-P(u)HQ(v))dudv.$$

On the other hand, $P^{\bar{k}}(p_{1,1})=\pi[k=\bar{k}|f(u)]=P^{\bar{k}}(u)$ and similarly $Q^{\bar{l}}(p_{2,1})=\pi[l=\bar{l}|g(v)]=Q^{\bar{l}}(v)$. Thus if $\Psi\in\mathcal{M}^{K,L}$ then $E[\Psi(P(p_{1,1}),Q(p_{2,1}))]=\int_0^1\int_0^1\Psi(P(u),Q(v))dudv$.
All this yields :

**Theorem 4.4.5**
*For all measurable function $\Psi : \Delta(K) \times \Delta(L) \to \mathbb{R}$, we have :*

$$\underline{T}(\Psi) = \sup_{(f,P)} \inf_{(g,Q)} F_1((f,P),(g,Q),\Psi) \tag{4.4.6}$$

$$\overline{T}(\Psi) = \inf_{(g,Q)} \sup_{(f,P)} F_1((f,P),(g,Q),\Psi) \tag{4.4.7}$$

*with*

$$\begin{aligned} F_1((f,P),(g,Q),\Psi) \quad := \int_0^1 \int_0^1 \Big\{ & 1\!\!1_{f(u)>g(v)}(P(u)HQ(v) - f(u)) \\ & + 1\!\!1_{f(u)<g(v)}(g(v) - P(u)HQ(v)) \\ & + \Psi(P(u),Q(v)) \Big\}\, dudv, \end{aligned} \tag{4.4.8}$$

*where $(f,P)$ satisfies to (4.4.4), and $(g,Q)$ satisfies to (4.4.5).*

### 4.4.4 Auxiliary recursive operators

Let us introduce two auxiliary recursive operators $T^1$ and $T^2$ on $\mathcal{M}^{K,L}$ corresponding to an auxiliary game with smaller strategy spaces : namely, the strategies are just the functions $P$ and $Q$.

$$T^1(\Psi)(p,q) := \max_{P\in\mathcal{P}(p)} \min_{Q\in\mathcal{Q}(q)} \int_0^1 \int_0^1 sg(u-v)P(u)HQ(v) + \Psi(P(u),Q(v))dudv \tag{4.4.9}$$

$$T^2(\Psi)(p,q) := \min_{Q\in\mathcal{Q}(q)} \max_{P\in\mathcal{P}(p)} \int_0^1 \int_0^1 sg(u-v)P(u)HQ(v) + \Psi(P(u),Q(v))dudv \tag{4.4.10}$$

where $\mathcal{P}(p)$ and $\mathcal{Q}(q)$ are defined in equation (4.3.1).
In this section, we will analyze this auxiliary game. Theorems 4.4.6 and 4.4.7 indicate that $T^1$ and $T^2$ map $\mathcal{L}^{K,L}$ on itself and that they coincide on this space. The remaining part of this subsection is devoted to the proof of lemma 4.4.14 that gives technical property of the optimal strategies $P^*$ and $Q^*$ in $T^1(\Psi)$ and $T^2(\Psi)$ that will be used in the next subsection to compare $T^1$, $T^2$ to $\underline{T}$, $\overline{T}$ : for appropriate $f^*$ and $g^*$ the pairs $(f^*,P^*)$ and $(g^*,Q^*)$ will be optimal strategies in $\underline{T}$ and $\overline{T}$.

**Theorem 4.4.6** *For all $\Psi \in \mathcal{L}^{K,L}$, the game corresponding to $T^1$, $T^2$ has a value (i.e. $T^1(\Psi) = T^2(\Psi)$) and both players have optimal strategies ($P^*$ for player 1 and $Q^*$ for player 2).*

**Proof** : The set $\mathcal{P}(p)$ and $\mathcal{Q}(q)$ are convex and compact for the weak* topology of $L^2$. Furthermore, since $\Psi$ is Lipschitz, for a fixed $Q$, the payoff function in the game is clearly continuous in $P$ for the strong topology of $L^2$. Due to the convexity of $\Psi$ in $P$, it is therefore also continuous for the weak* topology. Since a similar argument holds for $Q$, we may apply Sion's theorem.$\square$

**Theorem 4.4.7** *For any $\Psi \in \mathcal{L}^{K,L}$, $T^1(\Psi)$ and $T^2(\Psi)$ also belongs to $\mathcal{L}^{K,L}$.*

**Proof** : The proof is split is various steps. Let us first define a distance on the strategies space.

**Definition 4.4.8**
*Let $D^K(p,\tilde{p})$ be the Hausdorff distance between $\mathcal{P}(p)$ and $\mathcal{P}(\tilde{p})$ defined by*

$$D^K(p,\tilde{p}) = \max(d^K(p,\tilde{p}), d^K(\tilde{p},p))$$

*with*

$$d^K(p,\tilde{p}) := \max_{P\in\mathcal{P}(p)} \min_{\tilde{P}\in\mathcal{P}(\tilde{p})} \sum_{k\in K} E[|P^k - \tilde{P}^k|]$$

*where the expectation is taken considering that $P$ and $\tilde{P}$ are function of a uniform random variable $u$ on $[0,1]$.*
*Similarly, $D^L(q,\tilde{q})$ is the Hausdorff distance between $\mathcal{Q}(q)$ and $\mathcal{Q}(\tilde{q})$, we get*

$$D^L(q,\tilde{q}) = \max(d^L(q,\tilde{q}), d^L(\tilde{q},q))$$

*with*

$$d^L(q,\tilde{q}) := \max_{Q\in\mathcal{Q}(q)} \min_{\tilde{Q}\in\mathcal{Q}(\tilde{q})} \sum_{l\in L} E[|Q^l - \tilde{Q}^l|]$$

First, we prove a Lipschitz property, associated to Hausdorff distance, for the function $T^1(\Psi)$.

**Lemma 4.4.9** $\forall \Psi \in \mathcal{L}^{K,L}$, $\exists C \in \mathbb{R}^+$, $\forall p,\tilde{p} \in \Delta(K)$, $\forall q,\tilde{q} \in \Delta(L)$

$$|T^1(\Psi)(p,q) - T^1(\Psi)(\tilde{p},\tilde{q})| \leq C(D^K(p,\tilde{p}) + D^L(q,\tilde{q}))$$

Indeed, by the definition of the operator $T^1$, there exists $C_1 \in \mathbb{R}^+$ such that the difference between the first stage payoff verify the following inequality

$$\begin{array}{cl} & \int_0^1\int_0^1 sg(u-v)P(u)HQ(v)dudv - \int_0^1\int_0^1 sg(u-v)\tilde{P}(u)H\tilde{Q}(v)dudv \\ = & \int_0^1\int_0^1 sg(u-v)(P(u)-\tilde{P}(u))HQ(v)dudv + \int_0^1\int_0^1 sg(u-v)\tilde{P}(u)H(Q(v)-\tilde{Q}(v))dudv \\ \leq & C_1(\sum_{k\in K} E[|P^k - \tilde{P}^k|] + \sum_{l\in L} E[|Q^l - \tilde{Q}^l|]) \end{array}$$

By assumption $\Psi \in \mathcal{L}^{K,L}$, so there exists $C_2 \in \mathbb{R}^+$ such that for all $P,\tilde{P},Q,\tilde{Q}$,

$$|E[\Psi(P,Q) - \Psi(\tilde{P},\tilde{Q})]| \leq C_2(\sum_{k\in K} E[|P^k - \tilde{P}^k|] + \sum_{l\in L} E[|Q^l - \tilde{Q}^l|])$$

The definition of $T^1$ allows us to conclude with $C := C_1 + C_2$. $\square$
We next have to link the distances $D^K$ and $D^L$ to the norm on $\Delta(K) \times \Delta(L)$; the following lemma gives the result

**Lemma 4.4.10** *For all $p, \tilde{p} \in \Delta(K)$ and $q, \tilde{q} \in \Delta(L)$,*

$$D^K(p,\tilde{p}) = \sum_{k\in K} |p^k - \tilde{p}^k|, \quad D^L(q,\tilde{q}) = \sum_{l\in L} |q^l - \tilde{q}^l|$$

The proof of his lemma is given in the appendix.
So, the two previous lemmas give that for all $\Psi \in \mathcal{L}^{K,L}$ there exists $C \in \mathbb{R}^+$ such that

$$|T^1(\Psi)(p,q) - T^1(\Psi)(\tilde{p},\tilde{q})| \leq C\|(p,q) - (\tilde{p},\tilde{q})\|$$

Thus $T^1(\Psi)$ is in $\mathcal{L}^{K,L}$.$\square$

In order to compare $T^1$, $T^2$ with $\underline{T}$ and $\overline{T}$, we now need some results on $P^*$ and $Q^*$. Equations (4.4.16) and (4.4.19) are central to prove lemma 4.4.12.

Let $\overline{\Psi}(Q) := \int_0^1 \Psi(P^*(u), Q)du$ and define $\overline{\mathcal{R}}$ as

$$\overline{\mathcal{R}}(v) := \int_0^1 sg(u-v)P^*(u)Hdu = pH - 2\int_0^v P^*(u)duH \tag{4.4.11}$$

Since $(P^*, Q^*)$ is an equilibrium, $Q^*$ must be a best reply to $P^*$ in (4.4.9), so it must be optimal in the next minimization problem

$$T^1(\Psi)(p,q) = \min_{Q \underset{a.s.}{\in} \Delta(L), E[Q]=q} \int_0^1 \langle \overline{\mathcal{R}}(v), Q(v)\rangle + \overline{\Psi}(Q(v))dv \tag{4.4.12}$$

This minimum may clearly be replaced by

$$T^1(\Psi)(p,q) = \min_{Q \underset{a.s.}{\in} \Delta(L)} \sup_{x\in\mathbb{R}^L} \langle x, q - \int_0^1 Q(v)dv\rangle + \int_0^1 \langle \overline{\mathcal{R}}(v), Q(v)\rangle + \overline{\Psi}(Q(v))dv$$

This is a new game with the needed properties on the strategy spaces to apply Sion's theorem : this game has a value and $Q^*$ is an optimal strategy in that game. In particular,

$$T^1(\Psi)(p,q) = \sup_{x\in\mathbb{R}^L} \langle x, q\rangle + \min_{Q \underset{a.s.}{\in} \Delta(L)} \int_0^1 \langle \overline{\mathcal{R}}(v) - x, Q(v)\rangle + \overline{\Psi}(Q(v))dv \tag{4.4.13}$$

Since there is no constraint on the expectation of $Q$, a best reply $Q$ against $x$ must be such that for almost every $v$, $Q(v)$ minimizes $\langle \overline{\mathcal{R}}(v) - x, Q(v)\rangle + \overline{\Psi}(Q(v))$. Let us introduce the Fenchel conjugate $\overline{\Psi}^*$ of $\overline{\Psi}$.

$$\overline{\Psi}^*(y) := \inf_{z\in\Delta(L)} \langle y, z\rangle + \overline{\Psi}(z)$$

With this definition, (4.4.13) can be written as

$$T^1(\Psi)(p,q) = \sup_{x\in\mathbb{R}^L}\left(\langle x,q\rangle + \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x)dv\right) \tag{4.4.14}$$

The question we address now is that of the existence of an optimal $x$ :

**Lemma 4.4.11** *There exists $x^*$ optimal for the maximization problem (4.4.14)*

**Proof** : Let us define the convex function $A$ which worth $+\infty$ out of the simplex $\Delta(L)$ and, for any $\tilde{q}$ in $\Delta(L)$

$$A(\tilde{q}) := \sup_{x\in\mathbb{R}^L}\left(\langle x,\tilde{q}\rangle + \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x)dv\right)$$

Going backwards through the previous step up to equation (4.4.12) with $\tilde{q}$ instead of $q$ (In particular, $P^*$ used in the definitions of $\overline{\mathcal{R}}(v)$ and $\overline{\Psi}^*$ is still optimal in $T^1(\Psi)(p,q)$ and not in $T^1(\Psi)(p,\tilde{q})$.), we find that

$$A(\tilde{q}) = \min_{Q \underset{a.s.}{\in} \Delta(L), E[Q]=\tilde{q}} \int_0^1 \langle\overline{\mathcal{R}}(v), Q(v)\rangle + \overline{\Psi}(Q(v))dv$$

With a similar argument as in the proof of lemma 4.4.9 but with $P^*$ fixed, we get that $A$ is a Lipschitz function on $\Delta(L)$.
An $x$ that solves maximization problem (4.4.14) is simply an $x$ belongs to subgradient[1] $\check{\partial}A(\tilde{q})$. So, we just need to prove that, for all $\tilde{q}\in\Delta(L)$, the set $\check{\partial}A(\tilde{q})$ is non empty. But this property clearly holds for all $\tilde{q}\in\Delta(L)$ since $A$ is Lipschitz on its domain $\Delta(L)$.□

Let $x^*$ as in lemma 4.4.11, so, finally we obtain

$$T^1(\Psi)(p,q) = \langle x^*,q\rangle + \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x^*)dv \tag{4.4.15}$$

Since $Q^*$ must be a best reply against $x^*$ in (4.4.13), for almost all $v$, $Q^*(v)$ must belong to the supergradient[2] $\hat{\partial}$ of $\overline{\Psi}^*$ :

$$\text{For almost every } v : Q^*(v) \in \hat{\partial}\overline{\Psi}^*(\overline{\mathcal{R}}(v)-x^*) \tag{4.4.16}$$

In the same way, we can deal with $P^*$. Let us define

$$\underline{\Psi}(P) := \textstyle\int_0^1 \Psi(P,Q^*(v))dv$$
$$\underline{\Psi}_*(y) := \sup_{z\in\Delta(K)}\langle y,z\rangle + \underline{\Psi}(z)$$

[1] $\check{\partial}A(\tilde{q}) := \{y|\forall q,\ A(q)-A(\tilde{q}) \geq \langle y, q-\tilde{q}\rangle\}$
[2] $\hat{\partial}(\overline{\Psi}^*)(z) := \{y|\forall x,\ \overline{\Psi}^*(x)-\overline{\Psi}^*(z) \leq \langle y, x-z\rangle\}$

$$\underline{\mathcal{R}}(u) := \int_0^1 sg(u-v)HQ^*(v)dv = 2H\int_0^u Q^*(v)dv - Hq \tag{4.4.17}$$

By the same argument used to prove the existence of $x^*$, we have the existence of $y^*$ that is optimal in the minimization problem :

$$\inf_{y\in\mathbb{R}^K} \langle p,y\rangle + \int_0^1 \underline{\Psi}_*(\underline{\mathcal{R}}(u)-y)du$$

Then :

$$T^2(\Psi)(p,q) = \langle p,y^*\rangle + \int_0^1 \underline{\Psi}_*(\underline{\mathcal{R}}(u)-y^*)du \tag{4.4.18}$$

We also find that for almost all $u$, $P^*(u)$ must belong to the subgradient $\check{\partial}$ of $\underline{\Psi}_*$ :

$$\text{For almost every } u : P^*(u) \in \check{\partial}\underline{\Psi}^*(\underline{\mathcal{R}}(u)-y^*) \tag{4.4.19}$$

We will now take benefit of equations (4.4.16) and (4.4.19) to prove the following lemma.

**Lemma 4.4.12** *The function $t \to P^*(t)HQ^*(t)$ is almost surely equal to an increasing function.*

**Proof** : It is well known that the supergradient of a concave function $A$ is a decreasing correspondance : if $x \in \hat{\partial}A(y)$ and $x' \in \hat{\partial}A(y')$ then

$$\langle x-x', y-y'\rangle \le 0.$$

From equation (4.4.16), we find that for almost every $t,t' \in [0,1]$

$$\langle Q^*(t)-Q^*(t'), \overline{\mathcal{R}}(t)-\overline{\mathcal{R}}(t')\rangle \le 0$$

replacing $\overline{\mathcal{R}}$ by its definition 4.4.11 as an integral

$$\int_{t'}^t P^*(u)duH(Q^*(t)-Q^*(t')) \ge 0 \tag{4.4.20}$$

The same argument applies to equation (4.4.19) and leads to

$$(P^*(t)-P^*(t'))H\int_{t'}^t Q^*(u)du \ge 0 \tag{4.4.21}$$

Next, for $\epsilon > 0$, we define $F_\epsilon(s)$ (for $s \in [0,1-\epsilon]$) as

$$\frac{1}{\epsilon^2}\int_s^{s+\epsilon} P^*(u)duH\int_s^{s+\epsilon} Q^*(v)dv$$

We now observe that, up to a factor $\epsilon^{-2}$, the derivative $\frac{d}{ds}F_\epsilon(s)$ is just the sum of the left hand sides of the two previous inequalities evaluated at $t = s+\epsilon$ and $t' = s$. As a consequence, for almost every $s$, $\frac{d}{ds}F_\epsilon(s)$ is positive, so $F_\epsilon$ is almost surely equal to an increasing function.
Finally, since $\frac{1}{\epsilon}\int_s^{s+\epsilon} P^*(u)du$ (resp. $\frac{1}{\epsilon}\int_s^{s+\epsilon} Q^*(v)dv$) converge in $L^1$ to $P^*(s)$ (resp. $Q^*(s)$) as $\epsilon$ goes to 0, we get the almost sure convergence of $F_\epsilon$ to the function $t \to P^*(t)HQ^*(t)$.□

We conclude this section by proving that optimal $(P^*, Q^*)$ can be find such that $P^*$ and $Q^*$ are constant on each interval on which $P^*HQ^*$ are constant. We start by the following lemma

**Lemma 4.4.13** *If $P^*HQ^*$ is constant on the interval $[a,b]$, then there exist $P^\bullet$ and $Q^\bullet$ which verify*

1. *$P^\bullet$ and $Q^\bullet$ are constant on $[a,b]$.*
2. *$P^\bullet = P^*$ and $Q^\bullet = Q^*$ on the complementary of $[a,b]$.*
3. *$\int_0^1 P^\bullet(u)du = p$ and $\int_0^1 Q^\bullet(v)dv = q$.*
4. *$P^\bullet$ and $Q^\bullet$ are respectively optimal in $T^1$ and $T^2$.*
5. *$P^*HQ^* = P^\bullet HQ^\bullet$.*

**Proof** : Let us define $P^\bullet$ and $Q^\bullet$,
- $P^\bullet = P^*$ on $[0,1]\backslash[a,b]$ and $P^\bullet(t) = \frac{1}{b-a}\int_a^b P^*(u)du$ on $[a,b]$.
- $Q^\bullet = Q^*$ on $[0,1]\backslash[a,b]$ and $Q^\bullet(s) = \frac{1}{b-a}\int_a^b Q^*(v)dv$ on $[a,b]$.

So point (1), (2) and (3) are obvious and we have to prove now (4) and (5). We start with point (5) : since $P^*HQ^*$ is constant on $[a,b]$, inequalities (4.4.20) and (4.4.21) used to prove the increasing property of $P^*HQ^*$ are in fact equalities, so for any $s$ and $t$ in $[a,b]$,

$$\overline{\Psi}^*(\overline{\mathcal{R}}(s) - x^*) + \langle \overline{\mathcal{R}}(t) - \overline{\mathcal{R}}(s), Q^*(s)\rangle = \overline{\Psi}^*(\overline{\mathcal{R}}(t) - x^*) \tag{4.4.22}$$

In particular, the derivative with respect to $t$ of the previous equation gives,

$$P^*(t)HQ^*(s) = P^*(a)HQ^*(a) \tag{4.4.23}$$

In turn, this leads to, for all $t \in [a,b]$

$$P^\bullet(t)HQ^\bullet(t) = P^*(a)HQ^*(a) = P^*(t)HQ^*(t)$$

Furthermore, this equality must also hold outside of $[a,b]$ according to point (2). We prove now that $P^\bullet$ is optimal in $T^1$.
Let us define $\mathcal{R}^\bullet(v) := \int_0^1 sg(u-v)P^\bullet(u)Hdu$. The constant value of $P^\bullet$ has been

chosen in such a way that $\mathcal{R}^\bullet$ and $\overline{\mathcal{R}}$ coincide on the complementary of $[a,b]$. We now prove that

$$\int_a^b \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x^*)dv \leq \int_a^b \overline{\Psi}^*(\mathcal{R}^\bullet(v)-x^*)dv \tag{4.4.24}$$

Equations (4.4.22) and (4.4.23) give, for all $t$ in $[a,b]$,

$$\overline{\Psi}^*(\overline{\mathcal{R}}(a)-x^*) - 2(t-a)P^*(a)HQ^*(a) = \overline{\Psi}^*(\overline{\mathcal{R}}(t)-x^*)$$
$$\overline{\Psi}^*(\overline{\mathcal{R}}(b)-x^*) - 2(t-b)P^*(a)HQ^*(a) = \overline{\Psi}^*(\overline{\mathcal{R}}(t)-x^*)$$

Furthermore, after summation and integration in $t$ between $a$ and $b$ of the two previous equations, we get

$$\int_a^b \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x^*)dv = \frac{b-a}{2}\left(\overline{\Psi}^*(\overline{\mathcal{R}}(a)-x^*) + \overline{\Psi}^*(\overline{\mathcal{R}}(b)-x^*)\right)$$

Since $\mathcal{R}^\bullet$ is linear on $[a,b]$ and coincide with $\overline{\mathcal{R}}$ at the extreme points of the interval, we find that

$$\mathcal{R}^\bullet(t) = \frac{t-a}{b-a}\overline{\mathcal{R}}(b) + (1-\frac{t-a}{b-a})\overline{\mathcal{R}}(a)$$

So, the concavity of $\overline{\Psi}^*$ gives, for all $t$ in $[a,b]$

$$\overline{\Psi}^*(\mathcal{R}^\bullet(t)-x^*) \geq \frac{t-a}{b-a}\overline{\Psi}^*(\overline{\mathcal{R}}(b)-x^*) + (1-\frac{t-a}{b-a})\overline{\Psi}^*(\overline{\mathcal{R}}(a)-x^*)$$

The integral of this on $[a,b]$ yields equation (4.4.24) follows. Since $\mathcal{R}^\bullet$ and $\overline{\mathcal{R}}$ coincide on the complementary of $[a,b]$, we get

$$\langle x^*,q\rangle + \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(v)-x^*)dv \leq \langle x^*,q\rangle + \int_0^1 \overline{\Psi}^*(\mathcal{R}^\bullet(v)-x^*)dv$$

On the other hand, $\Psi$ is a concave function in $p$, and $P^\bullet$ may be viewed as a conditional expectation of $P^*$ (namely conditional to the variable $u \times 1\!\!1_{[a,b]^c}(u)$), so with Jensen's inequality we conclude that

$$\overline{\Psi}(Q(v)) \leq \int_0^1 \Psi(P^\bullet(u),Q(v))du$$

so, next

$$\begin{array}{rcl}
T^1(\Psi) & \leq & \langle x^*,q\rangle + \int_0^1 \overline{\Psi}^*(\mathcal{R}^\bullet(v)-x^*)dv \\
& \leq & \langle x^*,q\rangle + \min_{Q \underset{a.s.}{\in} \Delta(L)} \int_0^1 \langle \mathcal{R}^\bullet(v)-x^*, Q(v)\rangle + (\int_0^1 \Psi(P^\bullet(u),Q(v))du)dv \\
& \leq & \sup_x \min_{Q \underset{a.s.}{\in} \Delta(L)} \langle x,q\rangle + \int_0^1\int_0^1 \langle \mathcal{R}^\bullet(v)-x, Q(v)\rangle + \Psi(P^\bullet(u),Q(v))dudv \\
& \leq & \min_{Q \underset{a.s.}{\in} \Delta(L), E[Q]=q} \int_0^1\int_0^1 sg(u-v)P^\bullet(u)HQ(v) + \Psi(P^\bullet(u),Q(v))dudv
\end{array}$$

So, $P^\bullet$ guarantees $T^1(\Psi)$ to player 1 in the initial game defining $T^1$, and it is thus an optimal strategy. Since, the same argument holds for $Q^\bullet$ the lemma is proved. $\square$

Repeating recursively the modification of previous lemma on the sequence of the disjoint intervals of constance of $P^*HQ^*$ ranked by decreasing length, we get in the limit, optimal strategies $P^*$ and $Q^*$ that satisfy the following lemma :

**Lemma 4.4.14** *There exists a pair of optimal strategies $(P^*, Q^*)$ in $T^1(\Psi)$ and $T^2(\Psi)$ such that :*
*If $P^*(t)HQ^*(t) = P^*(s)HQ^*(s)$ then $P^*(t) = P^*(s)$ and $Q^*(t) = Q^*(s)$.*

In the following, $P^*$ and $Q^*$ are supposed to follow this property.

### 4.4.5 Relations between operators

In this section, we will provide optimal strategies for $\underline{T}$ and $\overline{T}$ based on the optimal $P^*$ and $Q^*$ of last section.

**Definition 4.4.15** *Let $\Psi \in \mathcal{L}^{K,L}$. Let $P^*$ and $Q^*$ be the optimal strategies in $T^1(\Psi)(p,q)$ and $T^2(\Psi)(p,q)$ as in lemma 4.4.14. We define $f^*$ and $g^*$ as*

$$f^*(u) = g^*(u) := \frac{1}{u^2}\int_0^u 2sP^*(s)HQ^*(s)ds. \tag{4.4.25}$$

The central point of this section is the following theorem :

**Theorem 4.4.16** *The pairs $(f^*, P^*)$ and $(g^*, Q^*)$ satisfy (4.4.4) and (4.4.5), furthermore,*

1. *$(f^*, P^*)$ guarantees $T^1(\Psi)(p,q)$ to player 1 in the definition of $\underline{T}(\Psi)(p,q)$ given in (4.4.6).*
2. *$(g^*, Q^*)$ guarantees $T^2(\Psi)(p,q)$ to player 2 in the definition of $\overline{T}(\Psi)(p,q)$ given in (4.4.7).*

Before dealing with the proof of this theorem, let us observe that it has as corollary :

**Corollary 4.4.17** *$T^2(\Psi)(p,q) = \overline{T}(\Psi)(p,q) = \underline{T}(\Psi)(p,q) = T^1(\Psi)(p,q)$ and thus $(f^*, P^*)$ and $(g^*, Q^*)$ are respectively optimal strategies in $\underline{T}(\Psi)(p,q)$ and $\overline{T}(\Psi)(p,q)$.*

Indeed, (1) and (2) in theorem 4.4.16 indicate respectively that

$$\underline{T}(\Psi)(p,q) \geq T^1(\Psi)(p,q) \text{ and } T^2(\Psi)(p,q) \geq \overline{T}(\Psi)(p,q)$$

Since, $\underline{T}(\Psi)(p,q) \leq \overline{T}(\Psi)(p,q)$, the result follows from theorem 4.4.6 that claims : $T^1(\Psi)(p,q) = T^2(\Psi)(p,q)$.□

**Proof of theorem 4.4.16** : The proof is based on various steps : we start with the following lemma :

**Lemma 4.4.18** *$f^*$ is $[0,1]$-valued, increasing. Furthermore, if $f^*(t_1) = f^*(t_2)$ with $t_1 < t_2$ then both $f^*$ and $P^*$ are constant on $[0,t_2]$. In particular, $(f^*, P^*)$ and $(g^*, Q^*)$ are strategies verifying (4.4.4) and (4.4.5).*

**Proof** : The elements of the matrix $H$ are supposed to be in $[0,1]$, so, since $P^*HQ^*$ is increasing, we conclude with equation (4.4.25) that

$$0 \leq f^*(u) \leq P^*(u)HQ^*(u) \leq 1 \tag{4.4.26}$$

Differentiating equation (4.4.25), we get the following differential equation

$$uf^{*'}(u) + 2f^*(u) = 2P^*(u)HQ^*(u) \tag{4.4.27}$$

With (4.4.26), we infer that $uf^{*'}(u) \geq 0$. So, $f^*$ is $[0,1]$-valued and increasing. Next, if $f^*(t_1) = f^*(t_2)$ with $0 \leq t_1 < t_2 \leq 1$. Then $f^*$ must be constant on the whole interval $[t_1,t_2]$. Therefore, $f^{*'}(t) = 0$ for $t$ in $[t_1,t_2]$. Thus by equations (4.4.27) with $u = t_2$ and (4.4.25), for any $t$ in $[t_1,t_2]$,

$$P^*(t_2)HQ^*(t_2) = f^*(t_2) = \frac{1}{t_2^2}\int_0^{t_2} 2sP^*(s)HQ^*(s)ds$$

So, we have

$$\frac{1}{t_2^2}\int_0^{t_2} 2s\left(P^*(t_2)HQ^*(t_2) - P^*(s)HQ^*(s)\right)ds = 0$$

Since $P^*HQ^*$ is increasing, this an integral of a positive function, so $P^*(s)HQ^*(s) = P^*(t_2)HQ^*(t_2)$ for all $s$ in the interval $[0,t_2]$. Finally, by lemma 4.4.14 and equation (4.4.25), the result follows : $f^*$ and $P^*$ are constant on $[0,t_2]$. □

Let start with a technical lemma

**Lemma 4.4.19** *If $\phi$ is a concave function on $\mathbb{R}^K$ and $v$, $z$ are bounded $\mathbb{R}^K$-valued measurable functions such that for almost every $t$ in $[0,1]$,*

$$z(t) \in \hat{\partial}\phi(\int_0^t v(s)ds)$$

*then for any $a$ and $b$ in $[0,1]$,*

$$\phi(b) - \phi(a) = \int_a^b \langle z(t), v(t)\rangle dt$$

**Proof :** Let us define for all $t$ in $[0,1]$, $x(t) := \int_0^t v(s)ds$, and

$$\begin{array}{l} F_\epsilon(t) := \frac{1}{\epsilon}(x(t+\epsilon)-x(t)) \\ G_\epsilon(t) := \frac{1}{\epsilon}(x(t)-x(t-\epsilon)) \end{array}$$

Furthermore, both $F_\epsilon$ and $G_\epsilon$ are converging almost surely to $v$. The dominated convergence theorem indicates then that :

$$\lim_{\epsilon\to 0}\int_a^b \langle z(t), F_\epsilon(t)\rangle dt = \int_a^b \langle z(t), v(t)\rangle dt = \lim_{\epsilon\to 0}\int_a^b \langle z(t), G_\epsilon(t)\rangle dt$$

Furthermore, the concavity of $\phi$ gives

$$\phi(x(t+\epsilon)) - \phi(x(t)) \le \langle z(t), x(t+\epsilon)-x(t)\rangle = \epsilon\langle z(t), F_\epsilon(t)\rangle$$

So, by integration on $[a,b]$, we get

$$\frac{1}{\epsilon}\int_b^{b+\epsilon}\phi(x(t))dt - \frac{1}{\epsilon}\int_a^{a+\epsilon}\phi(x(t))dt = \frac{1}{\epsilon}\left(\int_{a+\epsilon}^{b+\epsilon}\phi(x(t))dt - \int_a^b \phi(x(t))dt\right) \le \int_a^b \langle z(t), F_\epsilon(t)\rangle dt$$

Thus, as $\epsilon$ goes to 0, we obtain

$$\phi(b) - \phi(a) \le \int_a^b \langle z(t), x(t)\rangle dt$$

In the same way, we get :

$$\phi(x(t-\epsilon)) - \phi(x(t)) \le \langle z(t), x(t-\epsilon)-x(t)\rangle = \epsilon\langle z(t), G_\epsilon(t)\rangle$$

This reverse inequality leads us to the result.□

**Lemma 4.4.20** *For all $\alpha \in [0,1]$,*

$$\overline{\Psi}^*(\overline{\mathcal{R}}(\alpha) - x^*) + \alpha f^*(\alpha) - \int_\alpha^1 f^*(u)du = \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

*with $x^*$ defined in lemma 4.4.11.*

**Proof** : Let us define $S(u) := \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)$ and observe, according to lemma 4.4.19 and equations (4.4.16) and (4.4.11), that

$$S(1) - S(\alpha) = 2\int_1^\alpha P^*(s)HQ^*(s)ds$$

So, by integration of equation (4.4.27) between 1 and $\alpha$, we get

$$\alpha f^*(\alpha) - \int_\alpha^1 f^*(u)du - f^*(1) = S(1) - S(\alpha)$$

Equation (4.4.25) gives $f^*(1) = \int_0^1 2uP^*(u)HQ^*(u)du = -S(1) + \int_0^1 S(u)du$, so

$$S(\alpha) + \alpha f^*(\alpha) - \int_\alpha^1 f^*(u)du = \int_0^1 S(u)du$$

□

We now will prove assertion (1) in theorem 4.4.16. Let $A$ the payoff guaranteed by $(f^*, P^*)$ in $\underline{T}(\Psi)(p,q)$ (see formula (4.4.6)). So :

$$A := \inf_{(g,Q)} F_1((f^*, P^*), (g,Q), \Psi)$$

where $(g,Q)$ verifies (4.4.5), in particular $\int_0^1 Q(v)dv = q$, and $F_1$ defined as in equation (4.4.8). We have to prove that $A \geq T^1(\Psi)$.
With, as in previous section : $\overline{\Psi}(Q) := \int_0^1 \Psi(P^*(u), Q)du$, we get

$$\begin{aligned} F_1((f^*, P^*), (g,Q), \Psi) \quad := \quad & \int_0^1 \left\{ \left( \int_0^1 sg(f^*(u) - g(v))P^*(u)Hdu \right) Q(v) + \overline{\Psi}(Q(v)) \right\} dv \\ & + \int_0^1 \int_0^1 1\!\!1_{f^*(u)<g(v)} g(v) - 1\!\!1_{f^*(u)>g(v)} f^*(u) \; dudv \end{aligned}$$

In the above infimum, $(g,Q)$ are supposed to fulfill the three conditions of (4.4.5). We decrease the value of this infimum by dispensing $(g,Q)$ to fulfill the hypothesis c) in (4.4.5). Next, we may also dispense with the hypothesis b) that $\int_0^1 Q(v)dv = q$ by introducing a maximization over $x \in \mathbb{R}^L$ :

$$A \geq \inf_g \inf_{\substack{Q \in \Delta(L) \\ a.s.}} \sup_{x \in \mathbb{R}^L} \langle x, q - \int_0^1 Q(v)dv \rangle + F_1((f^*, P^*), (g,Q), \Psi)$$

where $Q$ is simply a $\Delta(L)$-valued mapping and $g$ an increasing $[0,1]$-valued function. So, since the $\inf \sup$ is always greater than the $\sup \inf$, we get

$$A \quad \geq \quad \sup_{x \in \mathbb{R}^L} \inf_g \inf_Q \langle x, q - \int_0^1 Q(v)dv \rangle + F_1((f^*, P^*), (g,Q), \Psi)$$

The expression we have to minimize in $(g,Q)$ is simply the expectation of some function $\int_0^1 \phi(g(v), Q(v))dv$. Optimal $(g,Q)$ can be find by taking constant functions $(g,Q)$ valued in

$$\mathop{argmin}_{g \in [0,1], Q \in \Delta(L)} \phi(g,Q).$$

Furthermore, the minimization over $Q$ will lead naturally to the function $\overline{\Psi}^*$ of last section. So, if we set :

$$B(x,g) := \overline{\Psi}^* \left( \int_0^1 sg(f^*(u) - g)P^*(u)Hdu - x \right) + \int_0^1 1\!\!1_{f^*(u)<g} g - 1\!\!1_{f^*(u)>g} f^*(u)du$$

we get :

$$\begin{aligned} A &\geq \sup_{x\in\mathbb{R}^L} \langle x, q\rangle + \inf_{g\in[0,1]} B(x,g) \\ &\geq \langle x^*, q\rangle + \inf_{g\in[0,1]} B(g) \end{aligned}$$

where $x^*$ was defined in lemma 4.4.11 and $B(g) := B(x^*, g)$.

Let us now observe that $f^*$ is increasing and continuous. The range of $f^*$ turns therefore to be an interval $[f^*(0), f^*(1)]$. Furthermore, according lemma 4.4.18, if we define $a = \sup\{u \in [0,1] | f^*(u) = f^*(0)\}$, we know that $f^*$ is constant on $[0, a]$ and strictly increasing on $[a, 1]$. The minimization on $g \in [0,1]$ can be split in four parts according to the shape of $f^*$ :

**Part 1) :** The minimization on $g$ in interval $]f^*(0), f^*(1)]$
**Part 2) :** The minimization on $g$ strictly less than $f^*(0)$.
**Part 3) :** The minimization on $g$ strictly greater than $f^*(1)$.
**Part 4) :** The minimization on $g = f^*(0)$.

We start with **part 1) :**
Any point $g$ in $]f^*(0), f^*(1)]$ can be written as $g = f^*(\alpha)$ with $\alpha \in ]a, 1]$. Since $f^*$ is strictly increasing on the interval $]a, 1]$,

$$sg(f^*(u) - g) = sg(u - \alpha)$$

and

$$1\!\!1_{f^*(u)<g}\, g - 1\!\!1_{f^*(u)>g} f^*(u) = 1\!\!1_{u<\alpha} f^*(\alpha) - 1\!\!1_{u>\alpha} f^*(u)$$

So, the argument of $\overline{\Psi}^*$ in $B(f^*(\alpha))$ is equal to the function $\overline{\mathcal{R}}(\alpha) - x^*$ where $\overline{\mathcal{R}}$ was defined in (4.4.11) and thus

$$B(g) = B(f^*(\alpha)) = \overline{\Psi}^*(\overline{\mathcal{R}}(\alpha) - x^*) + \alpha f^*(\alpha) - \int_\alpha^1 f^*(u)du$$

Therefore, with lemma 4.4.20, we get for all $g$ in $]f^*(0), f^*(1)]$ :

$$B(g) = \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

**Part 2) :** $(g < f^*(0))$
The argument of $\overline{\Psi}^*$ in $B(g)$ is just equal to $\int_0^1 P^*(u)Hdu - x^*$ and we get

$$B(g) = \overline{\Psi}^* \left(\overline{\mathcal{R}}(0) - x^*\right) - \int_0^1 f^*(u)du$$

So by lemma 4.4.20, we find that

$$B(g) = \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

**Part 3) :** $(g > f^*(1))$
The argument of $\overline{\Psi}^*$ in $B(g)$ is now $-\int_0^1 P^*(u)Hdu - x^*$ and with lemma 4.4.20, we get

$$B(g) = \overline{\Psi}^*(\overline{\mathcal{R}}(1) - x^*)du + g = \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du - f^*(1) + g$$

So, since $g > f^*(1)$, we get

$$B(g) > \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

**Part 4) :**$(g = f^*(0))$ In case of $a = 0$ then $f^*$ is strictly increasing on the whole interval $[0,1]$, so that the previous argument holds also in this case and

$$B(g) = \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

Next, if $a > 0$ then the argument of $\overline{\Psi}^*$ in $B(f^*(0))$ is $\int_a^1 P^*(u)Hdu - x^*$ and we get

$$B(f^*(0)) := \overline{\Psi}^*\left(\int_a^1 P^*(u)Hdu - x^*\right) - \int_a^1 f^*(u)du$$

Since $2\int_a^1 P^*(u)Hdu = \overline{\mathcal{R}}(a) + \overline{\mathcal{R}}(0)$, the concavity of $\overline{\Psi}^*$ gives,

$$\overline{\Psi}^*\left(\int_a^1 P^*(u)Hdu - x^*\right) \geq \frac{1}{2}\overline{\Psi}^*\left(\overline{\mathcal{R}}(a) - x^*\right) + \frac{1}{2}\overline{\Psi}^*\left(\overline{\mathcal{R}}(0) - x^*\right)$$

So by lemma 4.4.20, $\frac{1}{2}\overline{\Psi}^*\left(\overline{\mathcal{R}}(a) - x^*\right) + \frac{1}{2}\overline{\Psi}^*\left(\overline{\mathcal{R}}(0) - x^*\right)$ is equal to

$$\int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du + \int_a^1 f^*(u)du + \frac{1}{2}\left(\int_0^a f^*(u)du - af^*(a)\right)$$

Furthermore, $f^*$ is constant on the interval $[0,a]$, so $\int_0^a f^*(u)du - af^*(a) = 0$. Finally,

$$B(f^*(0)) \geq \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

So, all together, whatever the value of $g$ is, $B(g)$ is greater than

$$\int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du$$

and we conclude with equation (4.4.15), therefore, that

$$A \geq \langle x^*, q\rangle + \int_0^1 \overline{\Psi}^*(\overline{\mathcal{R}}(u) - x^*)du = T^1(\Psi).$$

Since, a similar argument holds for player 2, assertion (2) of theorem 4.4.16 is also true.□

We, now, apply inductively our results on the operators to prove the existence of $V_n$ :

**Theorem 4.4.21 (Existence of the value)**
*For all $n \in \mathbb{N}$, $V_{1,n} = V_{2,n} = V_n \in \mathcal{L}^{K,L}$ and $V_{n+1} = T^1(V_n) = T^2(V_n)$*

**Proof :** The result is obvious for $n = 0$. By induction, assume that the result holds for $n$. This implies that $V_{1,n} = V_{2,n} =: V_n$ is in $\mathcal{L}^{K,L}$. By hypothesis, $T^1(V_n) = T^2(V_n)$, so, due to the inequalities (3), (4) and proposition 4.4.2, $V_{1,n+1} \geq T^1(V_n) = T^2(V_n) \geq V_{2,n+1} \geq V_{1,n+1}$ , and thus by (2), $T^1(V_n) = T^2(V_n) = V_{2,n+1} = V_{1,n+1} \in \mathcal{L}^{K,L}$.□

# 4.5 The value

## 4.5.1 New formulation of the value

In this section, we want to provide a more tractable expression for the value $V_n$. We have $V_n = T^1(V_{n-1})$, so from now on : let us denote by $u_1$ and $v_1$ the uniform random variables appearing in the definition of $T^1(V_{n-1})$ and let also $P_1$ and $Q_1$ be the corresponding strategies. $P_1$ is $\sigma(u_1)$-measurable, $Q_1$ is $\sigma(v_1)$-measurable and we clearly have $E[P_1] = p$ and $E[Q_1] = q$. In the expression of $T^1(V_{n-1})$, we have to evaluate $V_{n-1}(P_1, Q_1)$ which in turn can be expressed as $T^1(V_{n-2})(P_1, Q_1)$. Let us denote by $u_2$ and $v_2$ the uniform random variables appearing in the definition of $T^1(V_{n-2})(P_1, Q_1)$ and let also $P_2$ and $Q_2$ be the corresponding strategies. So, $P_2$ now depends on $u_2$ and $u_1$, $v_1$ since it depends on $P_1$ and $Q_1$. Furthermore, $E[P_2|u_1, v_1] = P_1$ and $E[Q_2|u_1, v_1] = Q_1$.
Let then $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ be a system of independent random variables uniformly distributed on $[0, 1]$ and let us $G^1 := \{G^1_k\}_{k=1}^n$ and $G^2 := \{G^2_k\}_{k=1}^n$ as

$$G^1_k := \sigma(u_1, \ldots, u_k, v_1, \ldots, v_{k-1})$$

$$G_k^2 := \sigma(u_1, \ldots, u_{k-1}, v_1, \ldots, v_k)$$

Let also $\mathcal{G} := \{\mathcal{G}_k\}_{k=1}^n$ with $\mathcal{G}_k := \sigma(G_k^1, G_k^2)$.
So, applying the above proceeding recursively, we define $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$ and we get $P \in M_n^1(\mathcal{G}, p)$ and $Q \in M_n^2(\mathcal{G}, q)$ where :

**Definition 4.5.1**

1. *Let $M_n^1(\mathcal{G}, p)$ the set of $\Delta(K)$-valued $\mathcal{G}$-martingales $X = (X_1, \ldots, X_n)$ that are $G^1$-adapted and satisfying $E[X_1] = p$.*
2. *Similarly, let $M_n^2(\mathcal{G}, q)$ the set of all $\Delta(L)$-valued $\mathcal{G}$-martingales $Y = (Y_1, \ldots, Y_n)$ that are $G^2$-adapted and satisfying $E[Y_1] = q$.*

**Remark 4.5.2** *Let us observe that, if $X \in M_n^1(\mathcal{G}, p)$ and $Y \in M_n^2(\mathcal{G}, q)$, then the process $XHY := (X_1HY_2, \ldots, X_nHY_n)$ is also a $\mathcal{G}$-adapted martingale. Indeed,*

$$\begin{array}{rcl} E[X_{i+1}HY_{i+1}|\mathcal{G}_i] & = & E[E[X_{i+1}HY_{i+1}|G_{i+1}^1]|\mathcal{G}_i] \\ & = & E[X_{i+1}HE[Y_{i+1}|G_{i+1}^1]|\mathcal{G}_i] \end{array}$$

*Furthermore, $Y_{i+1}$ is $G_{i+1}^2$-measurable, so $Y_{i+1}$ is independent on $u_{i+1}$, and therefore*

$$E[Y_{i+1}|G_{i+1}^1] = E[Y_{i+1}|\mathcal{G}_i]$$

*So, we get*

$$\begin{array}{rcl} E[X_{i+1}HY_{i+1}|\mathcal{G}_i] & = & E[X_{i+1}HE[Y_{i+1}|\mathcal{G}_i]|\mathcal{G}_i] \\ & = & E[X_{i+1}|\mathcal{G}_i]HE[Y_{i+1}|\mathcal{G}_i] \\ & = & X_iHY_i \end{array}$$

With the previous definition, we obtain :

**Theorem 4.5.3** *For all $n \in \mathbb{N}$, for all $p \in \Delta(K)$ and $q \in \Delta(L)$, let $\underline{\mathcal{V}}_n(p,q)$ and $\overline{\mathcal{V}}_n(p,q)$ denote :*

$$\begin{array}{rcl} \underline{\mathcal{V}}_n(p,q) & := & \max_{P \in M_n^1(\mathcal{G},p)} \min_{Q \in M_n^2(\mathcal{G},q)} E[\sum_{i=1}^n sg(u_i - v_i) P_n H Q_n] \\ \overline{\mathcal{V}}_n(p,q) & := & \min_{Q \in M_n^2(\mathcal{G},q)} \max_{P \in M_n^1(\mathcal{G},p)} E[\sum_{i=1}^n sg(u_i - v_i) P_n H Q_n] \end{array}$$

*then*

$$V_n(p,q) = \underline{\mathcal{V}}_n(p,q) = \overline{\mathcal{V}}_n(p,q)$$

**Proof** : Sion's theorem can clearly by applied here and leads to $\underline{\mathcal{V}}_n = \overline{\mathcal{V}}_n$, so we have just to prove that

$$V_n \geq \underline{\mathcal{V}}_n \text{ and } \overline{\mathcal{V}}_n \geq V_n$$

We will now prove recursively the inequality $V_n \geq \underline{\mathcal{V}}_n$.
The formula holds for $n=0$, since $V_0 = 0 = \underline{\mathcal{V}}_0$.
Assume now that the result holds for $n$, then

$$V_{n+1}(p,q) \geq \max_{\{P \underset{a.s.}{\in} \Delta(K), \int_0^1 P(u)du=p\}} \min_{\{Q \underset{a.s.}{\in} \Delta(L), \int_0^1 Q(v)dv=q\}} B_n(P,Q)$$

where $B_n(P,Q) = \int_0^1\int_0^1 sg(u_1-v_1)P(u_1)HQ(v_1) + \underline{\mathcal{V}}_n(P(u_1),Q(v_1))du_1dv_1$.
Next observe that :

$$\underline{\mathcal{V}}_n(P(u_1),Q(v_1)) = \max_{\tilde{P}\in M_n^1(\mathcal{G},P(u_1))} \min_{\tilde{Q}\in M_n^2(\mathcal{G},Q(v_1))} E[\sum_{i=2}^{n+1} sg(u_i-v_i)\tilde{P}_{n+1}H\tilde{Q}_{n+1}]$$

Let us denote,

$$\begin{array}{l} \mathbf{M}^1_{n+1}(P) := \{\overline{P} \in M^1_{n+1}(\mathcal{G},p) | \forall u_1 \in [0,1], \overline{P}_1(u_1) = P(u_1)\} \\ \mathbf{M}^2_{n+1}(Q) := \{\overline{Q} \in M^2_{n+1}(\mathcal{G},q) | \forall v_1 \in [0,1], \overline{Q}_1(v_1) = Q(v_1)\} \end{array}$$

In particular, the sets $\mathbf{M}^1_{n+1}(P)$ and $\mathbf{M}^2_{n+1}(Q)$ are respectively subset of $M^1_{n+1}(\mathcal{G},p)$ and of $M^2_{n+1}(\mathcal{G},q)$. So, the process $\overline{P} := (P(u_1),\tilde{P}_2,\ldots,\tilde{P}_{n+1})$, with $\tilde{P}\in M^1_n(\mathcal{G},P(u_1))$ , belongs then obviously to $\mathbf{M}^1_{n+1}(P)$. However, it has the particularity that $\overline{P}_k$ is $(P(u_1),Q(v_1),u_2,\ldots,u_k,v_2,\ldots,v_k)$ measurable. The subset of $\mathbf{M}^1_{n+1}(P)$ of process with this last property will be denoted $\mathcal{M}^1_{n+1}(P,Q)$. Similarly, $\overline{Q} := (Q(v_1),\tilde{Q}_2,\ldots,\tilde{Q}_{n+1}) \in \mathbf{M}^2_{n+1}(Q)$ with for all $k$ :
$\overline{Q}_k$ is $(P(u_1),Q(v_1),u_2,\ldots,u_k,v_2,\ldots,v_k)$ measurable, we will denote by $\mathcal{M}^2_{n+1}(P,Q)$ the set of such processes. So, we get

$$B_n(P,Q) = \max_{\overline{P}\in\mathcal{M}^1_{n+1}(P,Q)} \min_{\overline{Q}\in\mathcal{M}^2_{n+1}(P,Q)} E[sg(u_1-v_1)\overline{P}_1H\overline{Q}_1 + \sum_{i=2}^{n+1} sg(u_i-v_i)\overline{P}_{n+1}H\overline{Q}_{n+1}] \tag{4.5.1}$$

Furthermore, since $(\overline{P}_kH\overline{Q}_k)_{k\geq 2}$ is a $\mathcal{G}$-martingale

$$\begin{array}{rcl} A(\overline{P},\overline{Q}) & := & E[sg(u_1-v_1)\overline{P}_1H\overline{Q}_1 + \sum_{i=2}^{n+1} sg(u_i-v_i)\overline{P}_{n+1}H\overline{Q}_{n+1}] \\ & = & E[sg(u_1-v_1)\overline{P}_1H\overline{Q}_1] + E[\sum_{i=2}^{n+1} sg(u_i-v_i)\overline{P}_iH\overline{Q}_i] \end{array}$$

So, if $\overline{P}$ is in $\mathbf{M}^1_{n+1}(P)$ and $\overline{Q}\in\mathcal{M}^2_{n+1}(P,Q)$ then, $\overline{Q}_i$ is $(P(u_1),Q(v_1),u_2,\ldots,u_i,v_2,\ldots,v_i)$-measurable, hence,

$$\begin{array}{rcl} A(\overline{P},\overline{Q}) & = & E[sg(u_1-v_1)\overline{P}_1H\overline{Q}_1] \\ & + & E[\sum_{i=2}^{n+1} sg(u_i-v_i)E[\overline{P}_i|P(u_1),Q(v_1),u_2,\ldots,u_i,v_2,\ldots,v_i]H\overline{Q}_i] \end{array}$$

So, the maximization over $\mathcal{M}^1_n(P,Q)$ in (4.5.1) is equal to the maximization over the set $\mathbf{M}^1_{n+1}(P)$ and since $\mathcal{M}^2_n(P,Q)\subset\mathbf{M}^2_{n+1}(Q)$ we get

$$\begin{array}{rcl} B_n(P,Q) & = & \max_{\overline{P}\in\mathbf{M}^1_{n+1}(P)} \min_{\overline{Q}\in\mathcal{M}^2_n(P,Q)} A(\overline{P},\overline{Q}) \\ & \geq & \max_{\overline{P}\in\mathbf{M}^1_{n+1}(P)} \min_{\overline{Q}\in\mathbf{M}^2_{n+1}(Q)} A(\overline{P},\overline{Q}) \end{array}$$

Moreover, according to remark 4.5.2, we have that

$$\begin{array}{rcl} E[sg(u_1-v_1)\overline{P}_1 H\overline{Q}_1] & = & E[sg(u_1-v_1)E[\overline{P}_{n+1}H\overline{Q}_{n+1}|\mathcal{G}_1]] \\ & = & E[sg(u_1-v_1)\overline{P}_{n+1}H\overline{Q}_{n+1}] \end{array}$$

So, $B_n$ satisfies to

$$B_n(P,Q) \geq \max_{\overline{P}\in\mathbf{M}^1_{n+1}(P)} \min_{\overline{Q}\in\mathbf{M}^2_{n+1}(Q)} E[\sum_{i=1}^{n+1} sg(u_i-v_i)\overline{P}_{n+1}H\overline{Q}_{n+1}]$$

Finally, $V_{n+1}(p,q)$ is greater than

$$\max_{\{P \underset{a.s.}{\in} \Delta(K), E[P]=p\}} \min_{\{Q \underset{a.s.}{\in} \Delta(L), E[Q]=q\}} \max_{\overline{P}\in\mathbf{M}^1_{n+1}(P)} \min_{\overline{Q}\in\mathbf{M}^2_{n+1}(Q)} E[\sum_{i=1}^{n+1} sg(u_i-v_i)\overline{P}_{n+1}H\overline{Q}_{n+1}]$$

Since $\min_Q \max_{\overline{P}}$ is obviously greater than the $\max_{\overline{P}} \min_Q$ and since the maximization over $(P,\overline{P})$ coincides with the maximization over the set $M^1_n(\mathcal{G},p)$, we get

$$V_{n+1}(p,q) \geq \max_{\overline{P}\in M^1_{n+1}(\mathcal{G},p)} \min_{\overline{Q}\in M^2_{n+1}(\mathcal{G},q)} E[\sum_{i=1}^{n+1} sg(u_i-v_i)\overline{P}_{n+1}H\overline{Q}_{n+1}]$$

The same way for the $\min\max$ problem provides the reverse inequality. This concludes the proof of the theorem.□

Remark 4.5.2 allows us to state the following corollary

**Corollary 4.5.4** *For all $p \in \Delta(K)$ and $q \in \Delta(L)$*

$$\begin{array}{rcl} V_n(p,q) & = & \max_{P\in M^1_n(\mathcal{G},p)} \min_{Q\in M^2_n(\mathcal{G},q)} E[\sum_{i=1}^n sg(u_i-v_i)P_iHQ_i] \\ & = & \min_{Q\in M^2_n(\mathcal{G},q)} \max_{P\in M^1_n(\mathcal{G},p)} E[\sum_{i=1}^n sg(u_i-v_i)P_iHQ_i] \end{array}$$

## 4.6 Asymptotic approximation of $V_n$

We aim to analyze in this paper the limit of $\frac{V_n}{\sqrt{n}}$. It is technically convenient to introduce here the quantity $W_n$ defined as

$$W_n(p,q) = \max_{P\in M^1_n(\mathcal{G},p)} \min_{Q\in M^2_n(\mathcal{G},q)} E[\sum_{i=1}^n 2(u_i-v_i)P_nHQ_n] \qquad (4.6.1)$$

As shown in the next theorem, there exists a constant $C$ independent on $n$ such that $\|V_n - W_n\|_\infty \leq C$. As a consequence, $\frac{V_n}{\sqrt{n}}$ and $\frac{W_n}{\sqrt{n}}$ will have the same limit.

**Theorem 4.6.1** *For all $p \in \Delta(K)$ and $q \in \Delta(L)$*

$$|V_n(p,q) - W_n(p,q)| \leq 2\|H\| \sqrt{\sum_k p^k(1-p^k) \sum_l q^l(1-q^l)}$$

*where* $\|H\| := \max_{\{x,y \neq 0\}} \frac{|xHy|}{\|x\|_2 \|y\|_2}$ *and* $\|p\|_2 := (\sum_{k \in K} |p^k|^2)^{\frac{1}{2}}$.

**Proof :** Let us fixe $P \in M_n^1(\mathcal{G}, p)$ and $Q \in M_n^2(\mathcal{G}, q)$. Corollary 4.5.4 leads us to compare $E[sg(u_i - v_i)P_i H Q_i]$ and $E[2(u_i - v_i)P_i H Q_i]$. We will now provide an upper bound on the difference of those two quantities. To simplify the formula, we set $S := sg(u_i - v_i)$, $\overline{S} := 2(u_i - v_i)$, $\Delta P := P_i - P_{i-1}$ and $\Delta Q := Q_i - Q_{i-1}$. Let us first observe that $E[S|G_i^1] = \int_0^1 sg(u_i - v_i)dv_i = 2u_i - 1 = E[\overline{S}|G_i^1]$ and similarly $E[S|G_i^2] = E[\overline{S}|G_i^2]$, furthermore $E[S|\mathcal{G}_i] = E[\overline{S}|\mathcal{G}_i] = 0$. In particular, we get that

$$E[S\ P_{i-1}HQ_{i-1}] = 0 = E[\overline{S}\ P_{i-1}HQ_{i-1}]$$

This leads to

$$E[S\ P_i H Q_i] = E[S\ \Delta P H Q_{i-1}] + E[S\ P_{i-1} H \Delta Q] + E[S\ \Delta P H \Delta Q] \qquad (4.6.2)$$

And the same equation holds with $\overline{S}$ instead of $S$. Next, since $\Delta P H Q_{i-1}$ is $G_i^1$-measurable and $P_{i-1}H\Delta Q$ is $G_i^2$-measurable, we obtain

$$E[S\ \Delta P H Q_{i-1}] = E[E[S|G_i^1]\ \Delta P H Q_{i-1}] = E[E[\overline{S}|G_i^1]\ \Delta P H Q_{i-1}] = E[\overline{S}\ \Delta P H Q_{i-1}]$$

$$E[S\ P_{i-1} H \Delta Q] = E[E[S|G_i^2]\ P_{i-1} H \Delta Q] = E[E[\overline{S}|G_i^2]\ P_{i-1} H \Delta Q] = E[\overline{S}\ P_{i-1} H \Delta Q]$$

Hence, equation (4.6.2) for $S$ and $\overline{S}$ gives

$$E[S\ P_i H Q_i] - E[\overline{S}\ P_i H Q_i] = E[(S - \overline{S})\ \Delta P H \Delta Q] \qquad (4.6.3)$$

Applying equation (4.6.3) for $i$ equal 1 to $n$, we get

$$\begin{aligned} A\ &:= |E[\textstyle\sum_{i=1}^n sg(u_i - v_i)P_i H Q_i] - E[\sum_{i=1}^n 2(u_i - v_i)P_i H Q_i]| \\ &= |E[\textstyle\sum_{i=1}^n (sg(u_i - v_i) - 2(u_i - v_i))(P_i - P_{i-1})H(Q_i - Q_{i-1})]| \\ &\leq 2\|H\| E[\textstyle\sum_{i=1}^n \|P_i - P_{i-1}\|_2 \|Q_i - Q_{i-1}\|_2] \end{aligned}$$

Moreover, by Cauchy schwartz inequality applied to the scalar product $(x,y) \to \sum_i x_i y_i$, we get

$$A \leq 2\|H\| E[\sqrt{\textstyle\sum_{i=1}^n \|P_i - P_{i-1}\|_2^2} \sqrt{\textstyle\sum_{i=1}^n \|Q_i - Q_{i-1}\|_2^2}]$$

Furthermore, the Cauchy schwartz inequality associated to the scalar product $(f,g) \to E[fg]$ gives

$$A \leq 2\|H\| \sqrt{E[\textstyle\sum_{i=1}^n \|P_i - P_{i-1}\|_2^2] E[\sum_{i=1}^n \|Q_i - Q_{i-1}\|_2^2]}$$

Since, for $i \neq j$, $E[\langle P_i - P_{i-1}, P_j - P_{j-1}\rangle] = 0$, we have

$$E[\sum_{i=1}^{n} \|P_i - P_{i-1}\|_2^2] = E[\|P_n - p\|_2^2]$$

and similarly for $Q$. It follows that

$$A \leq 2\|H\|\sqrt{E[\|P_n - p\|_2^2]E[\|Q_n - q\|_2^2]}$$

Furthermore, for any $k \in K$, $E[(P_n^k - p^k)^2] = E[(P_n^k)^2] - (p^k)^2 \leq p^k(1-p^k)$, thus we get

$$A \leq 2\|H\|\sqrt{\sum_k p^k(1-p^k)\sum_l q^l(1-q^l)}$$

Since the last equation is true for all pair of strategy $(P, Q)$, we get as announced that

$$|V_n(p,q) - W_n(p,q)| \leq 2\|H\|\sqrt{\sum_k p^k(1-p^k)\sum_l q^l(1-q^l)}$$

□

## 4.7 Heuristic approach to a continuous time game

We aim to analyze the limit of $\frac{V_n}{\sqrt{n}}$. However, we have no closed formula for $V_n$, as it was the case in the one sided information case. So, to analyze the asymptotic behavior of $\frac{V_n}{\sqrt{n}}$, we will have to provide a candidate limit $W^c$. Our aim is now to introduce a continuous time game, similar to the "Brownian games" introduced in [6], whose value would be $W^c$. As emphasized in the last section, $\frac{V_n}{\sqrt{n}}$ and $\frac{W_n}{\sqrt{n}}$ have the same asymptotic behavior, and the game $W^c$ appears more naturally with $W_n$. Indeed, according to equation (4.6.1), the random variables

$$S_k^{1,n} := \frac{\sqrt{3}}{\sqrt{n}}\sum_{i=1}^{k}(2u_i - 1) \text{ and } S_k^{2,n} := \frac{\sqrt{3}}{\sqrt{n}}\sum_{i=1}^{k}(2v_i - 1)$$

appear in the expression of $\sqrt{3}\frac{W_n}{\sqrt{n}}$ :

$$\sqrt{3}\frac{W_n}{\sqrt{n}}(p,q) = \max_{P \in M_n^1(\mathcal{G},p)} \min_{Q \in M_n^2(\mathcal{G},q)} E[(S_n^{1,n} - S_n^{2,n})P_n H Q_n]$$

Due to the Central Limit theorem, $S_k^{1,n}$ and $S_k^{2,n}$ converge in law to two independent standard normal $\mathcal{N}(0,1)$ random variables (This was the reason for the factor $\sqrt{3}$). In turn, those last random variables may be viewed as the value at 1 of two independent Brownian motions $\beta^1$ and $\beta^2$. To introduce $W^c$, the heuristic idea is to embed the martingale $P$ and $Q$ in the Brownian filtration and to see $P_n$ as a stochastic integrals :

$$P_n = p + \int_0^1 a_s d\beta_s^1 + \int_0^1 \bar{a}_s d\beta_s^2$$

Now, we have to express that $P_n$ is a $G^1$-adapted $\mathcal{G}$-martingale. In particular, $\Delta P := P_{i+1} - P_i$ is independent of $v_{i+1}$. $\Delta P$ is approximately equal to $a_s d\beta_s^1 + \bar{a}_s d\beta_s^2$ and $v_{i+1}$ equal to $d\beta_s^2$. So, $\bar{a}$ should be 0.
Furthermore, since $P_n$ belongs to $\Delta(K)$, the random variable $\int_0^1 a_s d\beta_s^1$ has finite variance, so that $\| \int_0^1 a_s d\beta_s^1 \|_{L^2}^2 = E[\int_0^1 a_s^2 ds] < +\infty$. This leads us to definitions 4.3.6 and 4.3.7 of the Brownian game $G^c(p,q)$ :

- The strategy space of player 1 is the set

$$\Gamma^1(p) := \left\{ (P_t)_{t\in\mathbb{R}^+} \;\middle|\; \begin{array}{l} \forall t \in \mathbb{R}^+,\ P_t \in \Delta(K), \exists a \in \mathcal{H}^2(\mathcal{F}) \\ \text{such that } P_t := p + \int_0^t a_s d\beta_s^1 \end{array} \right\}$$

- The strategy space of player 2 is the set

$$\Gamma^2(q) := \left\{ (Q_t)_{t\in\mathbb{R}^+} \;\middle|\; \begin{array}{l} \forall t \in \mathbb{R}^+,\ Q_t \in \Delta(L), \exists b \in \mathcal{H}^2(\mathcal{F}) \\ \text{such that } Q_t := q + \int_0^t b_s d\beta_s^2 \end{array} \right\}$$

- The payoff function of player 1 corresponding to a pair $P$, $Q$ is

$$E[(\beta_1^1 - \beta_1^2) P_1 H Q_1]$$

For a martingale $X$ on $\mathcal{F}$, we set

$$\|X\|_2 := \|X_\infty\|_{L^2} \tag{4.7.1}$$

The sets $\Gamma^1(p)$ and $\Gamma^2(q)$ are convex and bounded for the norm $\|.\|_2$, So they are compact for the weak* topology of $L^2$. Furthermore, since $E[(\beta_1^1 - \beta_1^2) P_1 H Q_1]$ is linear in $P$, for a fixed $Q$, the payoff function in the game is clearly continuous in $P$ for the strong topology of $L^2$. It is therefore also continuous for the weak* topology. Since a similar argument holds for $Q$, we may apply Sion's theorem to infer :

**Theorem 4.7.1** *For all $p \in \Delta(K)$ and $q \in \Delta(L)$, the game $G^c(p,q)$ has a value $W^c(p,q)$ :*

$$W^c(p,q) := \max_{P\in\Gamma^1(p)} \min_{Q\in\Gamma^2(q)} E[(\beta_1^1 - \beta_1^2) P_1 H Q_1] (= \min\max)$$

The next section is devoted to the comparison of $G_n(p,q)$ and $G^c(p,q)$.

## 4.8 Embedding of $G_n(p,q)$ in $G^c(p,q)$

We aim to prove that $\sqrt{3}\frac{W_n}{\sqrt{n}}$ converges to the value $W^c$ of the game $G^c(p,q)$. To this end, it will be useful to view $G_n(p,q)$ as a sub-game of $G^c(p,q)$, where players are restricted to smaller strategy spaces. More precisely, the game $G_n(p,q)$ is embedded in $G^c(p,q)$ as follows :
According to Azema-Yor (see [18]), there exists a $\mathcal{F}^1$-stopping time $T_1^n$ such that $\beta^1_{T_1^n}$ has the same distribution as $\frac{\sqrt{3}}{\sqrt{n}}(2u_1-1)$. In the same way, there exists a stopping time $\tau$ on the filtration $\sigma(\beta^1_{T_1^n+s}-\beta^1_{T_1^n}, s\leq t)$ such that $\frac{\sqrt{3}}{\sqrt{n}}(2u_2-1)$ has the same distribution as $\beta^1_{T_1^n+\tau}-\beta^1_{T_1^n}$. We write $T_2^n := T_1^n+\tau$. Doing this recursively, we obtain the following Skorohod's Embedding Theorem for the martingales $S^{1,n}$ and $S^{2,n}$. Furthermore, since $T_n^n$ is a sum of $n$ i.i.d random variables we may apply the law of large numbers to get in particular that $T_n^n$ converges to 1 in probability and the last part of the theorem can be found in [3].

**Theorem 4.8.1** *Let $\beta^1$ and $\beta^2$ be two independent Brownian motions and let $\mathcal{F}^1$ and $\mathcal{F}^2$ their natural filtrations. There exists a sequence of $0=T_0^n\leq\ldots\leq T_n^n$ of $\mathcal{F}^1$-stopping times such that the increments $T_k^n-T_{k-1}^n$ are independent, identically distributed, $E[T_k^n]=\frac{k}{n}<+\infty$ and for all $k\in\{0,\ldots,n\}$, $\beta^1_{T_k^n}$ has the same distribution as the random walk $S_k^{1,n}$.*
*There exists a similar sequence $0=R_0^n\leq\ldots\leq R_n^n$ of $\mathcal{F}^2$-stopping times such that the increments $R_k^n-R_{k-1}^n$ are independent, identically distributed, $E[R_k^n]=\frac{k}{n}<+\infty$ and for all $k\in\{0,\ldots,n\}$, $\beta^2_{R_k^n}$ has the same distribution as the random walk $S_k^{2,n}$.*
*Furthermore,*

$$\sup_{0\leq k\leq n}|T_k^n-\frac{k}{n}|\xrightarrow[n\to+\infty]{Prob}0 \text{ and } \sup_{0\leq k\leq n}|R_k^n-\frac{k}{n}|\xrightarrow[n\to+\infty]{Prob}0 \tag{4.8.1}$$

*As a consequence,*

$$\beta^1_{T_n^n}\xrightarrow[n\to+\infty]{L^2}\beta^1_1,\ \text{ and } \beta^2_{R_n^n}\xrightarrow[n\to+\infty]{L^2}\beta^2_1 \tag{4.8.2}$$

From now on, we will identify the random variables $\frac{\sqrt{3}}{\sqrt{n}}(2u_i-1)$ with $\beta^1_{T_i^n}-\beta^1_{T_{i-1}^n}$ and $\frac{\sqrt{3}}{\sqrt{n}}(2v_i-1)$ with $\beta^2_{R_i^n}-\beta^2_{R_{i-1}^n}$. Let us observe that for all $k$, the $\sigma$-algebra $G_k^1:=\sigma(u_1,\ldots,u_k,v_1,\ldots,v_{k-1})$ is a sub-$\sigma$-algebra of $\mathcal{F}^1_{T_k^n}\vee\mathcal{F}^2_{R_{k-1}^n}$ and similarly $G_k^2\subset\mathcal{F}^1_{T_{k-1}^n}\vee\mathcal{F}^2_{R_k^n}$, $\mathcal{G}_k\subset\mathcal{F}^1_{T_k^n}\vee\mathcal{F}^2_{R_k^n}$.

Let $P$ belongs to $M_n^1(\mathcal{G},p)$, $P_1$ as a function of $u_1$ is $\mathcal{F}^1_{T_1^n}$-measurable. It can be written as $P_1=p+\int_0^{T_1^n}a_s d\beta_s^1$, next, conditionally on $u_1,v_1$, $P_2$ is just a function of $u_2$ and thus $P_2-P_1$ may be written as $\int_{T_1^n}^{T_2^n}a_s d\beta^1_s$, where the process $a$ is

$\sigma(u_1, v_1, \beta^1_t, t \leq s)$-progressively measurable. Applying recursively this argument, we find that $P_n = p + \int_0^{T^n_n} a_s d\beta^1_s$, where $a_s 1\!\!1_{s\in[T^n_k, T^n_{k+1}[}$ is $\sigma(u_1, \ldots, u_k, v_1, \ldots, v_k, \beta^1_t, t \leq s)$-progressively measurable. It is convenient to define here $T^n_{n+1} = R^n_{n+1} = \infty$. With that convention, the process $a$ appearing above belongs to $\mathcal{H}^2_{1,n}$ where

$$\mathcal{H}^2_{1,n} := \left\{ a \,\middle|\, \begin{array}{l} \forall k \in \{0, \ldots, n\} : a_s 1\!\!1_{s\in[T^n_k, T^n_{k+1}[} \text{ is } \mathcal{F}^1_s \vee \mathcal{F}^2_{R^n_k} - \text{ prog. measurable} \\ \text{and } E[\int_0^\infty a_s^2 ds] < +\infty \end{array} \right\}$$

With this notation, we just have proved that if $P$ belongs to $M^1_n(\mathcal{G}, p)$ then $P_n$ is equal to $P_{T^n_n}$ for a process $P$ in $\Gamma^1_n(p)$, where :

$$\Gamma^1_n(p) := \left\{ (P_t)_{t\in\mathbb{R}^+} \,\middle|\, \begin{array}{l} \forall t \in \mathbb{R}^+,\ P_t \in \Delta(K), \exists a \in \mathcal{H}^2_{1,n} \\ \text{such that } P_t := p + \int_0^t a_s d\beta^1_s \end{array} \right\}$$

Similarly, if $Q$ in $M^2_n(\mathcal{G}, q)$, we may represent $Q_n$ as $q + \int_0^{R^n_n} b_s d\beta^2_s$, where $b_s 1\!\!1_{s\in[R^n_k, R^n_{k+1}[}$ is $\sigma(u_1, \ldots, u_k, v_1, \ldots, v_k, \beta^2_t, t \leq s)$-progressively measurable. The process $b$ belongs to $\mathcal{H}^2_{2,n}$ where

$$\mathcal{H}^2_{2,n} := \left\{ b \,\middle|\, \begin{array}{l} \forall k \in \{0, \ldots, n\} : b_s 1\!\!1_{s\in[R^n_k, R^n_{k+1}[} \text{ is } \mathcal{F}^1_{T^n_k} \vee \mathcal{F}^2_s - \text{ prog. measurable} \\ \text{and } E[\int_0^\infty b_s^2 ds] < +\infty \end{array} \right\}$$

Also if $Q$ belongs to $M^2_n(\mathcal{G}, q)$ then $Q_n$ is equal to $Q_{R^n_n}$ for a process $Q$ in $\Gamma^2_n(p)$, where :

$$\Gamma^2_n(q) := \left\{ (Q_t)_{t\in\mathbb{R}^+} \,\middle|\, \begin{array}{l} \forall t \in \mathbb{R}^+,\ Q_t \in \Delta(L), \exists b \in \mathcal{H}^2_{2,n} \\ \text{such that } Q_t := q + \int_0^t b_s d\beta^2_s \end{array} \right\}$$

Now, observe that $\Gamma^1_n(p)$ is in fact broader than $M^1_n(\mathcal{G}, p)$, and similarly, for $\Gamma^2_n(q)$. It is convenient to introduce here an extended game $G^c_n(p, q)$, where strategy spaces are respectively $\Gamma^1_n(p)$ and $\Gamma^2_n(q)$. The next theorem indicates that this extended game has the same value as $G_n(p, q)$ :

**Theorem 4.8.2** *For all $p \in \Delta(K)$ and $q \in \Delta(L)$,*

$$\sqrt{3}\frac{W_n}{\sqrt{n}}(p, q) = \max_{P\in\Gamma^1_n(p)} \min_{Q\in\Gamma^2_n(q)} E[(\beta^1_{T^n_n} - \beta^2_{R^n_n}) P_{T^n_n} H Q_{R^n_n}] \tag{4.8.3}$$

**Proof :** Let us define $\sqrt{3}\frac{\widetilde{W}_n}{\sqrt{n}}$ as the right hand side in formula (4.8.3) and let also introduce $\sqrt{3}\frac{W^\wedge_n}{\sqrt{n}}$ and $\sqrt{3}\frac{W^\vee_n}{\sqrt{n}}$ as

$$\sqrt{3}\frac{W^\wedge_n}{\sqrt{n}} := \max_{P\in M^1_n(\mathcal{G},p)} \min_{Q\in\Gamma^2_n(q)} E[(\beta^1_{T^n_n} - \beta^2_{R^n_n}) P_n H Q_{R^n_n}]$$

$$\sqrt{3}\frac{W_n^{\vee}}{\sqrt{n}} := \min_{Q\in M_n^2(\mathcal{G},q)} \max_{P\in\Gamma_n^1(p)} E[(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_{T_n^n} H Q_n]$$

Due to the compactness of $\Delta(K)$ and $\Delta(L)$, $\Gamma_n^1(p)$ and $\Gamma_n^2(q)$ are compact convex set for the weak* topology of $L^2$, so, Sion's theorem indicates that max and min commute in the previous equations. So, we will prove that $\widetilde{W}_n = W_n$ by proving that

$$\widetilde{W}_n \geq W_n^{\wedge} = W_n = W_n^{\vee} \geq \widetilde{W}_n$$

Since, $M_n^1(\mathcal{G},p)$ is included in $\Gamma_n^1(p)$, the first inequality is obvious from the definitions of $\widetilde{W}_n$ and $W_n^{\wedge}$. The other inequality follows from the fact that $M_n^2(\mathcal{G},q)$ is included in $\Gamma_n^2(q)$ and the definitions of $W_n^{\vee}$ and $\widetilde{W}_n$ as min-max. The equality $W_n^{\wedge} = W_n$ follows from next lemma that indicates that if $Q$ belongs to $\Gamma_n^2(q)$ then $(\overline{Q}_k)_{k=1,\dots,n}$ belongs to $M_n^2(\mathcal{G},q)$ where $\overline{Q}_k := E[Q_{R_n^n}|\mathcal{G}_k]$. Indeed, whenever $P$ is in $M_n^1(\mathcal{G},p)$, $(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_n H$ is $\mathcal{G}_n$-measurable, therefore

$$E[(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_n H Q_{R_n^n}] = E[(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_n H \overline{Q}_n]$$

As a consequence,

$$\min_{Q\in\Gamma_n^2(q)} E[(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_n H Q_{R_n^n}] = \min_{\overline{Q}\in M_n^2(\mathcal{G},q)} E[(\beta^1_{T_n^n} - \beta^2_{R_n^n})P_n H \overline{Q}_n]$$

And $W_n^{\wedge} = W_n$ as announced. The proof of $W_n = W_n^{\vee}$ is similar.□

**Lemma 4.8.3** *If $Q$ belongs to $\Gamma_n^2(q)$ then $(\overline{Q}_k)_{k=1,\dots,n}$ belongs to $M_n^2(\mathcal{G},q)$ where $\overline{Q}_k := E[Q_{R_n^n}|\mathcal{G}_k]$.*

**Proof :** Let $Q$ in $\Gamma_n^2(q)$. Then $Q_t = q + \int_0^t b_s d\beta_s^2$ for a process $b$ in $\mathcal{H}^2_{2,n}$. Obviously, $(\overline{Q}_k)_{k=1,\dots,n}$ is a $\mathcal{G}$-martingale and

$$Q_{R_k^n} - Q_{R_{k-1}^n} = \int_0^{R_k^n} 1\!\!1_{[R_{k-1}^n, R_k^n[}(s) b_s d\beta_s^2. \tag{4.8.4}$$

Since $b_s 1\!\!1_{s\in[R_{k-1}^n,R_k^n[}$ is $\mathcal{F}^1_{T_{k-1}^n} \vee \mathcal{F}^2_s$- progressively measurable, $Q_{R_k^n} - Q_{R_{k-1}^n}$ is $\mathcal{F}^1_{T_{k-1}^n} \vee \mathcal{F}^2_{R_k^n}$-measurable. Next, $u_k$ is independent on $\mathcal{F}^1_{T_{k-1}^n} \vee \mathcal{F}^2_{R_k^n}$, so in particular,

$$E[Q_{R_k^n} - Q_{R_{k-1}^n}|\mathcal{G}_k] = E[Q_{R_k^n} - Q_{R_{k-1}^n}|\sigma(G_k^2, u_k)] = E[Q_{R_k^n} - Q_{R_{k-1}^n}|G_k^2]$$

Now, let us observe that $Q_{R_{k-1}^n}$ is $\mathcal{F}^1_{T_{k-1}^n} \vee \mathcal{F}^2_{R_{k-1}^n}$-measurable, thus, since $u_k$ and $v_k$ are independent of $\mathcal{F}^1_{T_{k-1}^n} \vee \mathcal{F}^2_{R_{k-1}^n}$, we have $\overline{Q}_{k-1} = E[Q_{R_{k-1}^n}|\mathcal{G}_k]$. Finally, equation (4.8.4) gives

$$\overline{Q}_k = E[Q_{R_k^n}|\mathcal{G}_k] = \overline{Q}_{k-1} + E[Q_{R_k^n} - Q_{R_{k-1}^n}|G_k^2]$$

And $\overline{Q}_k$ is then $G_k^2$-measurable.□

## 4.9 Convergence of $G_n^c(p,q)$ to $G^c(p,q)$

Our aim in this section is to prove the following theorem

**Theorem 4.9.1** *$\sqrt{3}\frac{W_n}{\sqrt{n}}$ converges uniformly to $W^c$.*

The proof of this result is based on two following approximations results for strategies in continuous game by strategies in $G_n^c(p,q)$. The proof of these lemmas is a bit technical and will be postponed to the next section.

**Lemma 4.9.2** *let $P^*$ be an optimal strategy of player 1 in $G^c(p,q)$, there exists a sequence $P^n$ in $\Gamma_n^1(p)$ converging to $P^*$ with respect to the norm $\|.\|_2$ defined in (4.7.1). Similarly, if $Q^*$ is an optimal strategy of player 2 in $G^c(p,q)$, there exists a sequence $Q^n$ in $\Gamma_n^2(q)$ converging to $Q^*$.*

and

**Lemma 4.9.3** *Let $\alpha$ be an increasing mapping from $\mathbb{N}$ to $\mathbb{N}$ and $Q^{\alpha(n)}$ be a strategy of player 2 in $G^c_{\alpha(n)}(p,q)$ such that $Q^{\alpha(n)}_{R^{\alpha(n)}_{\alpha(n)}}$ converges for the weak* topology of $L^2$ to $Q$. Then $Q_t := E[Q|\mathcal{F}_{t\wedge 1}]$ is a strategy of player 2 in $G^c(p,q)$.*

**Proof of theorem 4.9.1 :**
Let $P^*$ be an optimal strategy of player 1 in $G^c(p,q)$ and $P^n$ as in lemma 4.9.2. Since, $(\beta^1_{T^n_n}-\beta^2_{R^n_n})HQ_{R^n_n}$ is bounded in $L^2$, the strategy $P^n$ guarantees, in $G_n^c(p,q)$ the amount

$$\sqrt{3}\frac{W_n}{\sqrt{n}}(p,q) \geq \min_{Q\in\Gamma_n^2(q)} E[(\beta^1_{T^n_n}-\beta^2_{R^n_n})P_1^*HQ_{R^n_n}] - C\|P^n_{T^n_n}-P_1^*\|_{L^2}$$

where $C$ is independent on $n$. Next,

$$\|P^n_{T^n_n}-P_1^*\|_{L^2} \leq \|P^n_{T^n_n}-P^*_{T^n_n}\|_{L^2} + \|P^*_{T^n_n}-P_1^*\|_{L^2} \leq \|P^n-P^*\|_2 + \|P^*_{T^n_n}-P_1^*\|_{L^2}$$

Since $P^*$ is a continuous martingale bounded in $L^2$, we get with equation 4.8.1 that $\|P^*_{T^n_n}-P^*_1\|_{L^2}$ converges to 0. Due to lemma 4.9.2, $\|P^n-P^*\|_2$ converges also to 0. Finally, with equation 4.8.2,

$$\sqrt{3}\tfrac{W_n}{\sqrt{n}}(p,q) \quad \geq \quad \min_{Q\in\Gamma_n^2(q)} E[(\beta^1_1-\beta^2_1)P_1^*HQ_{R^n_n}] - \epsilon_n$$

with $\epsilon_n \underset{n\to+\infty}{\longrightarrow} 0$.
Now, if $Q^n$ is optimal in last minimization problem, we get

$$\sqrt{3}\tfrac{W_n}{\sqrt{n}}(p,q) \quad \geq \quad E[(\beta^1_1-\beta^2_1)P_1^*HQ^n_{R^n_n}] - \epsilon_n \tag{4.9.1}$$

Let $\alpha$ be non decreasing function $\mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to+\infty} E[(\beta_1^1 - \beta_1^2)P_1^* H Q^{\alpha(n)}_{R^{\alpha(n)}_{\alpha(n)}}] = \liminf_{n\to+\infty} E[(\beta_1^1 - \beta_1^2)P_1^* H Q^n_{R^n_n}]$$

Since $Q^{\alpha(n)}$ is $\Delta(L)$-valued, by considering a subsequence, we may assume that $Q^{\alpha(n)}_{R^{\alpha(n)}_{\alpha(n)}}$ converges for the weak* topology of $L^2$ to a limit $Q$. So, lemma 4.9.3 may be applied and we get $Q_t = E[Q|\mathcal{F}_{t\wedge 1}]$ in $\Gamma^2(q)$.
Finally, since $E[(\beta_1^1 - \beta_1^2)P_1^* HQ]$ is a continuous linear functional of $Q$, we have

$$\lim_{n\to+\infty} E[(\beta_1^1 - \beta_1^2)P_1^* H Q^{\alpha(n)}_{R^{\alpha(n)}_{\alpha(n)}}] = E[(\beta_1^1 - \beta_1^2)P_1^* HQ] = E[(\beta_1^1 - \beta_1^2)P_1^* HQ_1]$$

$P^*$ being optimal in $G^c(p,q)$, we get with equation (4.9.1) :

$$\liminf_{n\to+\infty} \sqrt{3}\frac{W_n}{\sqrt{n}}(p,q) \geq E[(\beta_1^1 - \beta_1^2)P_1^* HQ_1] \geq W^c(p,q)$$

Symmetrically, the same argument for the player 2 provides the reverse inequality :

$$\limsup_{n\to+\infty} \sqrt{3}\frac{W_n}{\sqrt{n}}(p,q) \leq W^c(p,q)$$

Finally, for concave-convex function the point-wise convergence implies the uniform convergence (see [19]) and the theorem is proved.□

## 4.10 Approximation results

It will be convenient to introduce the random times $R^n(s)$. At time $s$ when playing in $G^c_n(p,q)$, player 1 knows $\beta^2_t$ for $t \leq R^n(s)$. Formally, $R^n(s)$ is defined as :

$$R^n(s) := \sum_{k=0}^{n} \mathbb{1}_{[T^n_k, T^n_{k+1}[}(s) R^n_k$$

In the following, we will say that an increasing mapping $\alpha : \mathbb{N} \to \mathbb{N}$ is a proper sequence if

$$\sup_{0\leq k\leq\alpha(n)} |T^{\alpha(n)}_k - \frac{k}{\alpha(n)}| \xrightarrow[n\to+\infty]{a.s.} 0 \text{ and } \sup_{0\leq k\leq\alpha(n)} |R^{\alpha(n)}_k - \frac{k}{\alpha(n)}| \xrightarrow[n\to+\infty]{a.s.} 0 \qquad (4.10.1)$$

With equation (4.8.1) in theorem 4.8.1, note that from any sequence, we may extract a proper subsequence.

This allows us to prove the next lemma :

**Lemma 4.10.1** *$R^n$ verifies the following properties :*

1. *For a fixed $s$, $R^n(s)$ is a stopping time on the filtration (in $t$) :*
$$(\mathcal{F}^1_s \vee \mathcal{F}^2_t)_{t\in\mathbb{R}^+}$$
2. *If $s \le t$ then $R^n(s) \le R^n(t)$.*
3. *If $\alpha$ is a proper subsequence, then for all $s \in [0,1]$, $R^{\alpha(n)}(s) \xrightarrow[n\to+\infty]{a.s.} s$.*

**Proof :** (2) is obvious since $R^n_k$ and $T^n_k$ are increasing sequences with $k$.
For fixed $t$, we have :

$$\{R^n(s) \le t\} = \cup_{k=0}^{n-1}\{T^n_k \le s < T^n_{k+1}\} \cap \{\ R^n_k \le t\}$$

Since $T^n_k$ is an $\mathcal{F}^1$-stopping time the set $\{T^n_k \le s < T^n_{k+1}\}$ belongs to $\mathcal{F}^1_s$ and similarly $R^n_k$ is an $\mathcal{F}^2$-stopping time so $\{R^n_k \le t\} \in \mathcal{F}^2_t$. As a consequence $\{R^n(s) \le t\}$ is in $\mathcal{F}^1_s \vee \mathcal{F}^2_t$, and (1) is proved.
Let $\alpha$ be a proper subsequence and let $s$ in $[0,1]$, let $\epsilon_n$ defined as

$$\epsilon_n := \max(\sup_{0\le k\le\alpha(n)} |T^{\alpha(n)}_k - \frac{k}{\alpha(n)}|, \sup_{0\le k\le\alpha(n)} |R^{\alpha(n)}_k - \frac{k}{\alpha(n)}|)$$

and let $k^n(s)$ in $\{1,\dots,\alpha(n)\}$ such that $R^{\alpha(n)}(s) = R^{\alpha(n)}_{k^n(s)}$ : we have

$$\frac{k^n(s)}{\alpha(n)} - \epsilon_n \le T^{\alpha(n)}_{k^n(s)} \le s < \min(T^{\alpha(n)}_{k^n(s)+1}, 1) \le \frac{k^n(s)+1}{\alpha(n)} + \epsilon_n$$

Therefore,

$$s - \frac{1}{\alpha(n)} - 2\epsilon_n \le \frac{k^n(s)}{\alpha(n)} - \epsilon_n \le R^{\alpha(n)}(s) = R^{\alpha(n)}_{k^n(s)} \le \frac{k^n(s)+1}{\alpha(n)} + \epsilon_n \le s + \frac{1}{\alpha(n)} + 2\epsilon_n$$

Since $\epsilon_n$ converges almost surely to 0, claim (3) is proved.□

**Lemma 4.10.2** *Let $a$ be in $\mathcal{H}^2(\mathcal{F})$. Then there exists a sequence $a^n$ in $\mathcal{H}^2_{1,n}$ such that $\|a^n - a\|_{\mathcal{H}^2}$ converges to 0.*

**Proof :** Let us first observe that the vector space generated by processes $a_s := 1\!\!1_{[t_1,t_2[}(s)\psi$ where $t_1 \le t_2$ belong to $[0,1]$ and $\psi$ is a bounded $\mathcal{F}_{t_1}$-measurable random variable is dense in $\mathcal{H}^2(\mathcal{F})$. So, it is just enough to prove the result for such processes $a$.
For a fixed $s \in \mathbb{R}^+$, $R^n(s)$ is a stopping time with respect to the filtration $(\mathcal{G}^s_t)_{t\ge0}$ where $\mathcal{G}^s_t := \mathcal{F}^1_s \vee \mathcal{F}^2_t$. The past $\mathcal{G}^s_{R^n(s)}$ of this filtration at $R^n(s)$ is thus well

defined.
Now let us define, for all $s$ and $n$,

$$a^n_s := 1\!\!1_{[t_1,t_2[}(s)\sum_{k=0}^{n} 1\!\!1_{[T^n_k,T^n_{k+1}[}(s)E[\psi|\mathcal{F}^1_s \vee \mathcal{F}^2_{R^n_k}]$$

We claim that $a^n$ is in $\mathcal{H}^2_{1,n}$.
Indeed, for fixed $n$, the process $X^k_s := E[\psi|\mathcal{F}^1_s \vee \mathcal{F}^2_{R^n_k}]$ is a martingale with respect to the continuous filtration $(\mathcal{F}^1_s \vee \mathcal{F}^2_{R^n_k})_{s\geq 0}$ and in particular, $X^k$ may be supposed càdlàg. Hence, the process $1\!\!1_{[T^n_k,T^n_{k+1}[}(s)a^n_s = 1\!\!1_{[t_1,t_2[}(s)1\!\!1_{[T^n_k,T^n_{k+1}[}(s)X^k_s$ is then $\mathcal{F}^1_s \vee \mathcal{F}^2_{R^n_k}$-progressively measurable. Furthermore, $\psi$ is in $L^2(\mathcal{F}_{t_1})$, so $a^n$ is then in $\mathcal{H}^2_{1,n}$.
Next, let us observe that for all $s$, $a^n_s = E[a_s|\mathcal{G}^s_{R^n(s)}]$ almost everywhere.
Indeed, for fixed $s$, let us first denote $Y_t := E[\psi|\mathcal{G}^s_t]$. $Y$ is a continuous bounded martingale with respect to the continuous filtration $(\mathcal{F}^1_s \vee \mathcal{F}^2_t)_{t\geq 0}$. So, stopping theorem applies and $E[\psi|\mathcal{G}^s_{R^n(s)}] = Y_{R^n(s)}$. In turn, due to the definition of $R^n(s)$, we get

$$\begin{aligned}
E[a_s|\mathcal{G}^s_{R^n(s)}] &= 1\!\!1_{[t_1,t_2[}(s)Y_{R^n(s)} \\
&= 1\!\!1_{[t_1,t_2[}(s)\textstyle\sum_{k=0}^{n} 1\!\!1_{[T^n_k,T^n_{k+1}[}(s)Y_{R^n_k} \\
&= 1\!\!1_{[t_1,t_2[}(s)\textstyle\sum_{k=0}^{n} 1\!\!1_{[T^n_k,T^n_{k+1}[}(s)X^k_s \\
&= a^n_s
\end{aligned}$$

Let next $\alpha$ be a proper subsequence, we now prove that :

$$\text{For all } s : a^{\alpha(n)}_s \text{ converges almost surely to } a_s. \tag{4.10.2}$$

Indeed, for $s > 1$, $a^n_s = 0 = a_s$. On the other hand, for $s$ in $[0,1]$, by point (3) in lemma 4.10.1, $R^{\alpha(n)}_s$ converges almost surely to $s$. Due to the continuity of $Y_t$, $Y_{R^{\alpha(n)}(s)}$ converges almost surely to $Y_s = E[\psi|\mathcal{F}_s]$. Finally, since $\psi$ is $\mathcal{F}_{t_1}$-measurable, we get $a^{\alpha(n)}_s$ almost surely converges to $1\!\!1_{[t_1,t_2[}(s)E[\psi|\mathcal{F}_s] = a_s$.
Since both $a^{\alpha(n)}_s$ and $a_s$ are bounded, we get successively with (4.10.2) and Lebesgue's dominated convergence theorem that : for all $s$, $E[(a^{\alpha(n)}_s - a_s)^2]$ converges to 0 and that $\|a^{\alpha(n)} - a\|_{\mathcal{H}^2} = \int_0^1 E[(a^{\alpha(n)}_s - a_s)^2]ds$ converges to 0.
We are now in position to conclude the proof : Wouldn't indeed $a^n$ converges to $a$, there would exist a subsequence $\gamma(n)$ and $\epsilon > 0$ such that for all $n$, $\|a^{\gamma(n)} - a\|_{\mathcal{H}^2} > \epsilon$. But, this is in contradiction with the fact that we may extract from $\gamma$ a proper subsequence $\alpha$ ($\alpha(\mathbb{N}) \subset \gamma(\mathbb{N})$) for which $\|a^{\alpha(n)} - a\|_{\mathcal{H}^2}$ converges to 0. □

**Proof of lemma 4.9.3 :**
Due to the previsible representation of the Brownian filtration, $Q_t$ may be written as $q + \int_0^t a_s d\beta^1_s + \int_0^t b_s d\beta^2_s$ with $a$ and $b$ in $\mathcal{H}^2(\mathcal{F})$. So to prove that $Q_t$ is

in $\Gamma^2(q)$, we just have to prove that the process $a$ is equal to 0. This can be demonstrated by proving that for all process $Y_t = \int_0^t y_s d\beta^1_s$ with $y$ in $\mathcal{H}^2(\mathcal{F})$, $E[Y_1 Q_1] = E[\int_0^1 a_s y_s ds] = 0$ .
From lemma 4.10.2, there exists $y^n$ in $\mathcal{H}^2_{1,n}$ such that $\|y^n - y\|_{\mathcal{H}^2}$ converges to 0. We set $Y^n_t := \int_0^t y^n_s d\beta^1_s$ and for all $k$ in $\{0, \dots, \alpha(n)\}$, $\overline{Y}^n_k := Y^{\alpha(n)}_{T^{\alpha(n)}_k}$ and $\overline{Q}^n_k := Q^{\alpha(n)}_{R^{\alpha(n)}_k}$. we get

$$\begin{aligned} \|\overline{Y}^n_{\alpha(n)} - Y_1\|_{L^2} &\leq \|\overline{Y}^n_{\alpha(n)} - Y_{T^{\alpha(n)}_{\alpha(n)}}\|_{L^2} + \|Y_1 - Y_{T^{\alpha(n)}_{\alpha(n)}}\|_{L^2} \\ &\leq \|y^{\alpha(n)} - y\|_{\mathcal{H}^2} + \|Y_1 - Y_{T^{\alpha(n)}_{\alpha(n)}}\|_{L^2} \end{aligned}$$

From equation (4.8.1) in theorem 4.8.1 and the continuity of $Y$, we infer that $\|\overline{Y}^n_{\alpha(n)} - Y_1\|_{L^2}$ converges to 0 and since $\overline{Q}^n_{\alpha(n)}$ is $\Delta(L)$-valued, we conclude that

$$E[\overline{Y}^n_{\alpha(n)}\overline{Q}^n_{\alpha(n)} - Y_1\overline{Q}^n_{\alpha(n)}] \underset{n\to+\infty}{\longrightarrow} 0$$

The weak* convergence of $\overline{Q}^n_{\alpha(n)}$ to $Q$ implies $E[Y_1\overline{Q}^n_{\alpha(n)}] \underset{n\to+\infty}{\longrightarrow} E[Y_1 Q]$ and so,

$$E[\overline{Y}^n_{\alpha(n)}\overline{Q}^n_{\alpha(n)}] \underset{n\to+\infty}{\longrightarrow} E[Y_1 Q] = E[Y_1 Q_1]$$

Hence, the lemma follows at once if we prove that for all $n$, $E[\overline{Y}^n_{\alpha(n)}\overline{Q}^n_{\alpha(n)}] = 0$.
Let us first define for all $k \in \{1, \dots, \alpha(n)\}$,

$$\overline{G}^{1,n}_k := \mathcal{F}^1_{T^{\alpha(n)}_k} \vee \mathcal{F}^2_{R^{\alpha(n)}_{k-1}} \text{ and } \overline{G}^{2,n}_k := \mathcal{F}^1_{T^{\alpha(n)}_{k-1}} \vee \mathcal{F}^2_{R^{\alpha(n)}_k}$$

and for all $k \in \{0, \dots, \alpha(n)\}$,

$$\overline{\mathcal{G}}^n_k := \mathcal{F}^1_{T^{\alpha(n)}_k} \vee \mathcal{F}^2_{R^{\alpha(n)}_k}$$

Let us observe that $\overline{Y}^n_k$ is a $\overline{G}^{1,n}_k$-adapted $\overline{\mathcal{G}}^n_k$-martingale and $\overline{Q}^n_k$ is a $\overline{G}^{2,n}_k$-adapted $\overline{\mathcal{G}}^n_k$-martingale.
Furthermore, a similar argument as in remark 4.5.2 gives that the process $\overline{Y}^n_k\overline{Q}^n_k$ is a $(\overline{\mathcal{G}}^n_k)_{0\leq k\leq n}$-martingale. Hence, since $\overline{Y}^n_0 = Y^{\alpha(n)}_{T^{\alpha(n)}_0} = Y^{\alpha(n)}_0 = 0$, we get $E[\overline{Y}^n_{\alpha(n)}\overline{Q}^n_{\alpha(n)}] = E[\overline{Y}^n_0\overline{Q}^n_0] = 0$ and the lemma follows. □

**Proof of lemma 4.9.2 :**
Let us first remind that $P^*_t$ may be written as $p + \int_0^t a_s d\beta^1_s$ with $a$ in $\mathcal{H}^2(\mathcal{F})$. So, with lemma 4.10.2, we know that $a$ is the limit for the $\mathcal{H}^2$ norm of a sequence $\tilde{a}^n$ in $\mathcal{H}^2_{1,n}$. We set $\tilde{P}^n_t = p + \int_0^t \tilde{a}^n_s d\beta^1_s$. $\tilde{P}^n$ is not necessarily a strategy : it could exit the simplex $\Delta(K)$. To get rid of this problem, we proceed as follows :

First, observe that if, for some $k$, $p_k = 0$, then $(P^*)_k = 0$ almost surely. Therefore, there is no loss of generality in this case to assume that the $k$-th component of $\tilde{a}^n$ is equal to 0. The new sequence we would obtain by canceling the $k$-th component of $\tilde{a}^n$, would also converge to $a$. So, by reduction to a lower dimensional simplex, we may consider that $p_k > 0$, for all $k$. Let $\epsilon_n$ be a sequence of positive numbers such that

$$\frac{1}{\epsilon_n}\|\tilde{a}^n - a\|_{\mathcal{H}^2} \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } \epsilon_n \underset{n\to+\infty}{\longrightarrow} 0 \tag{4.10.3}$$

Let $\tau_n$ be the first time $p + (1-\epsilon_n)\int_0^t \tilde{a}^n_s d\beta^1_s$ exits the interior of the simplex $\Delta(K)$ and define $a^n_s := (1-\epsilon_n)1\!\!1_{s\leq\tau_n}\tilde{a}^n_s$. The process $P^n_t := p + \int_0^t a^n_s d\beta^1_s$ is now clearly a strategy of player 1 in $G^c_n(p,q)$, and

$$\begin{aligned}\|P^n - P^*\|_2 &= \|a^n - a\|_{\mathcal{H}^2}\\ &\leq \|a^n_. - (1-\epsilon_n)1\!\!1_{\leq\tau_n}a.\|_{\mathcal{H}^2} + (1-\epsilon_n)\|1\!\!1_{>\tau_n}a.\|_{\mathcal{H}^2} + \epsilon_n\|a\|_{\mathcal{H}^2}\end{aligned}$$

The last term in the last inequality tends clearly to 0 with $\epsilon_n$ since $a$ is in $\mathcal{H}^2(\mathcal{F})$. The first term is equal to $(1-\epsilon_n)\|1\!\!1_{.\leq\tau_n}(\tilde{a}^n_. - a.)\|_{\mathcal{H}^2} \leq (1-\epsilon_n)\|\tilde{a}^n - a\|_{\mathcal{H}^2}$ which converge to 0 according to the definitions of $\tilde{a}^n$. Furthermore, since $a_s = 0$ for $s > 1$, we have

$$\|1\!\!1_{.>\tau_n}a.\|^2_{\mathcal{H}^2} = E[\int_{\tau_n}^{\infty}(a_s)^2 ds] \leq E[1\!\!1_{1\geq\tau_n}\int_0^1 (a_s)^2 ds]$$

Furthermore, since $\xi := \int_0^1 (a_s)^2 ds$ is in $L^1$, $\{\xi\}$ is an uniformly integrable family. Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $A$ with $P(A) < \delta$ we have $E[1\!\!1_A\xi] \leq \epsilon$. So, in order to conclude that $\|P^n - P^*\|_2$ converge to 0, it just remains for us to prove that $P(1 \geq \tau_n)$ tends to 0.

Let us denote by $\Pi^n$ the homothety of center $p$ and ratio $\frac{1}{1-\epsilon_n}$. The distance between the complementary of $\Pi^n(\Delta(K))$ and $\Delta(K)$ is proportional to $\frac{\epsilon_n}{1-\epsilon_n}$. So, let $\eta > 0$ such that $d(\Delta(K), (\Pi^n(\Delta(K)))^c) = \frac{\epsilon_n}{1-\epsilon_n}\eta$ for all $n$.
Let us observe that if $\sup_{t\geq 0}|\tilde{P}^n_t - P^*_t| < \frac{\epsilon_n}{1-\epsilon_n}\eta$ then $\tau_n = +\infty$. Indeed, since $P^*$ is $\Delta(K)$-valued, we have that, for all $t$, $\tilde{P}^n_t \in \Pi^n(\Delta(K))$, and so for all $t$, $(\Pi^n)^{-1}(\tilde{P}^n_t) = p + (1-\epsilon_n)\int_0^t \tilde{a}^n_s d\beta^1_s \in \Delta(K)$. Hence, the definition of $\tau_n$ indicates that $\tau_n = +\infty$.
Hence, with Doob inequality, we get

$$P(1 \geq \tau_n) \leq P(\sup_{t\geq 0}|\tilde{P}^n_t - P^*_t| \geq \frac{\epsilon_n}{1-\epsilon_n}\eta) \leq 4(\frac{1-\epsilon_n}{\eta})^2\frac{1}{\epsilon_n^2}\|\tilde{P}^n - P^*\|_2^2$$

Finally, with equation (4.10.3) $P(1 \geq \tau_n)$ tends to 0 and the lemma follows.□

## 4.11 Appendix

**Proof of lemma 4.4.10 :**
We prove the following equality :
*For all* $p, \tilde{p} \in \Delta(K)$

$$d^K(p,\tilde{p}) = \sum_{k\in K} |p^k - \tilde{p}^k|$$

**Proof :** Let us remind that $\mathcal{P}(p) := \{P \underset{a.s.}{\in} \Delta(K),\ E[P] = p\}$, we get immediately the following inequality

$$\begin{array}{rcl} d^K(p,\tilde{p}) & \geq & \min_{\tilde{P}\in\mathcal{P}(\tilde{p})} \sum_{k\in K} E[|p^k - \tilde{P}^k|] \\ & \geq & \min_{\tilde{P}\in\mathcal{P}(\tilde{p})} \sum_{k\in K} |E[p^k - \tilde{P}^k]| \\ & \geq & \sum_{k\in K} |p^k - \tilde{p}^k| \end{array}$$

We next deal with the reverse inequality :
Let us fix $p$ in the simplex $\Delta(K)$ and $P$ in $\mathcal{P}(p)$. We have to prove that, for all $\tilde{p} \in \Delta(K)$

$$\left\{ \begin{array}{l} \textit{there exists } \tilde{P} \in \mathcal{P}(\tilde{p}) \textit{ such that for all } k \\ \\ \qquad E[|P^k - \tilde{P}^k|] = |p^k - \tilde{p}^k| \end{array} \right. \tag{4.11.1}$$

Let us define the hyperplane $\mathcal{H} := \{x \in \mathbb{R}^K | \sum_{i=1}^K x_i = 1\}$ in $\mathbb{R}^K$, so $\Delta(K) = [0,1]^K \cap \mathcal{H}$. Let us introduce a the covering of $[0,1]^K$ defined by the sets $C$ of the form $C = \Pi_{k=1}^K I_k$ where $I_k$ equal to $[0, p_k]$ or $[p_k, 1]$.
We will now work $C$ by $C$ and we prove that assertion (4.11.1) holds for all $\tilde{p} \in C \cap \mathcal{H}$. By reordering the coordinates, there is no loss of generality to assume that $C = C(p)$ with

$$C(p) := \Pi_{k=1}^l [0, p_k] \times \Pi_{k=l+1}^K [p_k, 1]$$

Let us define the set $B$,

$$B := \{\tilde{p} \in C(p) \cap \mathcal{H}, |\text{there exists } \tilde{P} \in \mathcal{P}(\tilde{p}) \text{ such that, } \tilde{P} \underset{a.s.}{\in} C(P)\}$$

Notice that, if $\tilde{p} \in B$ then there exists $\tilde{P} \in \mathcal{P}(\tilde{p})$ such that

$$E[|P^k - \tilde{P}^k|] = sign(p^k - \tilde{p}^k) E[P^k - \tilde{P}^k] = |p^k - \tilde{p}^k|$$

And (4.11.1) holds then for $\tilde{p}$. So, we have just to prove that, $C(p) \cap \mathcal{H} \subset B$.
Since $B$ is convex, it is sufficient to prove that : any extreme point $x$ of $C(p) \cap \mathcal{H}$ is in $B$.

Furthermore, extreme points $x$ of $C(p) \cap \mathcal{H}$ verify the following property :
There exists $m \in [1, K]$ such that

$$\begin{cases} x_m & \in & I_m \\ x_i & \in & \partial(I_i) \text{ , for } i \neq m \end{cases}$$

Let $x$ verifying these properties,
**case 1 :** There exists $k$ such that $x_k = 1$, thus

$$\tilde{P} \underset{a.s.}{=} x \underset{a.s.}{\in} \mathcal{P}(x) \text{ and obviously } \tilde{P} \underset{a.s.}{\in} C(P).$$

**case 2 :** Obviously, the case $x = p$ is ok.
**case 3 :** We now assume that, for all $i$, $x_i < 1$ and $x \neq p$.
First, according to the definition of $C(p)$ and $x$, we have $m > l$.
Indeed, if $m \leq l$ then $x_j = p^j$ for all $j > l$, so

$$x_m = 1 - \sum_{j \neq m} x_j = 1 - \sum_{j > l} p^j - \sum_{j \leq l, j \neq m} x_j$$

Furthermore, $x \neq p$, thus there exists $k \leq l$ such that $x_k < p^k$, so the definition of $I_j$ with $j \leq l$ leads us to

$$1 - \sum_{j > l} p^j - \sum_{j \leq l, j \neq m} x_j > 1 - \sum_{j > l} p^j - \sum_{j \leq l, j \neq m} p^j = p^m$$

so, we get the contradiction $x_m > p^m$ ($x_m \notin [0, p_m] = I_m$).
Furthermore, let $\tilde{P}$ such that

$$\begin{cases} \tilde{P}^i \underset{a.s.}{=} 0 & \text{for } i \leq l \text{ such that } x_i = 0 \\ \tilde{P}^i \underset{a.s.}{=} P^i & \text{for } i \neq m \text{ such that } x_i = p_i \\ \tilde{P}^m \underset{a.s.}{=} 1 - \sum_{i \neq m} \tilde{P}^i & \end{cases}$$

So, the previous definition gives, $\tilde{P}^m \underset{a.s.}{\geq} P^m$, $\tilde{P} \underset{a.s.}{\in} \mathcal{P}(x)$ and $\tilde{P} \underset{a.s.}{\in} C(P)$. The result follows.$\square$

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# Chapitre 5

# An algorithm to compute the value of Markov chain games

A. Marino

The recursive formula for the value of the zero-sum repeated games with incomplete information is frequently used to determine the value asymptotic behavior. Values of those games were linked to linear program analysis for a long time. The known approaches haven't any links with the recursive structure of the game and doesn't provide any explicit formula for the value. In this paper, we naturally connect the recursive operator and a parametric linear program. Furthermore, in order to determine recursively the game values, we provide an algorithm giving explicitly the value of such linear program. This proceeding is particularly useful in the framework of Markov chain games for which analysis of simple example has already shown the analysis difficulties. Finally, efficacy of our algorithm is verified on solved or unsolved examples.

## 5.1 Introduction

The origin of this paper is mainly due to the lack of intuition when we have to analyze repeated zero-sum games with lack of information. In this context, past literatures have typically analyzed the existence of value and optimal strategies for players. A number of papers underline the interest of analyzing the asymptotic behavior of the value, for example to make explicit the limit and the speed of convergence. In the repeated market games framework, see [2], De Meyer and Marino analyzed the value behavior and underline the usefulness to take an algorithmic approach. In this model, an algorithmic point of view seemed to be inevitable to intuitively infer the result. More generally, let us observe that the value analysis is straightforward related to the recursive structure of the game

and that the game recursive formula provides a good way for an algorithmic analysis. In this paper, we analyze repeated Markov chain games introduced in [1] by J. Renault. Those games provide a interesting framework for several reasons : In [1], J. Renault analyzes this repeated games and provides an underlying recursive formula linking values $V_n$ and $V_{n-1}$. Although J. Renault shows, in a theoretical way, the existence of the value and its limit, he provides a simple example for which the value and its asymptotic behavior are unknown.
In this paper, we approach algorithmically the recursive operator of a Markov chain games and we provide a process to determine explicitly the game value. In particular, this proceeding allows us to answer graphically to the previous problem and also to intuitively infer possible asymptotic results. This program may allow us to understand some problems which are apparently complex and to have an intuitive approach concerning the value and its asymptotic behavior.

This paper is split as explained below :
We first provide the entire description of a Markov chain game in the first section. Next, we remind the recursive structure of the game and we also give the recursive formula associated to the repeated game values. Furthermore, we connect this formula to a natural recursive operator and in section 5.4, we will observe that a parametric linear program appears naturally in our analysis. Hence, our problematic leads us to study an algorithmic approach for general parametric linear program in section 5.5. Sections 5.6 will be devoted to the induced results by the previous algorithm and will give several explanations concerning the implementation of our proceeding. Finally, the last section deals with several known examples and gives some details on program efficacy.

## 5.2 The model

First, we remind the model introduced by J.Renault in [1]. If $\mathcal{S}$ is a finite set, let us define $|\mathcal{S}|$ the cardinal of the set $\mathcal{S}$ and $\Delta(\mathcal{S})$ the set of probabilities on $\mathcal{S}$. $\Delta(\mathcal{S})$ will be naturally considered as a subset of $\mathbb{R}^{\mathcal{S}}$. Let us also denote by $K := \{1, \ldots, |K|\}$ the set of states of nature, $I$ the actions set of player 1 and $J$ those of player 2.

In the following, $K, I, J$ are supposed to be finite. In the development of the program, we will make the following additional assumption : The cardinal of $K$ is equal to 2. In the general description of the model, this hypothesis will not be considered. Now, we introduce a family of $|I| \times |J|$-payoff matrices for player 1 : $(G^k)_{k \in K}$, and a Markov chain on $K$ defined by an initial probability $p$ on $\Delta(K)$ and a transition matrix $M = (M_{kk'})_{(k,k') \in K \times K}$. All elements of $M$ are supposed to be positive and for all $k \in K$ : $\Sigma_{k'} M_{kk'} = 1$.

Moreover, an element $q$ in $\Delta(K)$ may be represented by a row vector $q = (q^1, \ldots, q^{|K|})$ with $q^k \geq 0$ for any $k$ and $\sum_{k \in K} q^k = 1$.

The Markov chain properties give in particular that, if $q$ is the law on the states of nature at some stage, the law at the next stage is then $qM$. We denote, for all $k \in K$, $\delta_k$ the Dirac measure on $k$.

The play of the zero-sum game proceeds in the following way :

- At the first stage, probability $p$ initially chooses a state $k_1$ and only player 1 is informed of $k_1$. Players 1 and 2 independently choose an action $i_1 \in I$ and $j_1 \in J$, respectively. The payoff of player 1 is then $G^{k_1}(i_1, j_1)$, and $(i_1, j_1)$ is publicly announced, and the game proceed to the next step.
- At stage $2 \leq q \leq n$, probability $\delta_{k_{q-1}} M$ chooses a state $k_q$, only player 1 is informed of this state. The players independently select an action in their own set of actions, $i_q$ and $j_q$ respectively. The stage payoff for player 1 is then $G^{k_q}(i_q, j_q)$, and $(i_q, j_q)$ is publicly announced, and the game proceed to the next stage.

Payoffs are not announced after each stage, players are assumed to have perfect recall and the whole description of the game is a public knowledge.

Now, we define the notion of behavior strategy in this game for player 1. A behavior strategy for player 1 is a sequence $\sigma = (\sigma_q)_{1 \leq q \leq n}$ where for all $n \geq 1$, $\sigma_q$ is a mapping from $(K \times I \times J)^{q-1} \times K$ to $\Delta(I)$. In other words, $\sigma_q$ generates a mixed strategy at stage $q$ depending on past and current states and past actions played. As we can see in the game description, states of nature are not available for player 2, so a behavior strategy for player 2 is a sequence $\tau = (\tau_q)_{1 \leq q \leq n}$, where for all $q$, $\tau_q$ is defined as a mapping from the cartesian product $(I \times J)^{n-1}$ to $\Delta(J)$. In the following, we denote by $\Sigma$ and $\mathcal{T}$, respectively, the set of behavior strategies of player 1 and player 2. According to $p$, a strategy profile $(\sigma, \tau)$ induces naturally a probability on $(K \times I \times J)^n$, and we denote $\gamma_n^p$ the expected payoff for player 1 :

$$\gamma_n^p(\sigma, \tau) := E_{p,\sigma,\tau}[\sum_{q=1}^{N} G^{k_q}(i_q, j_q)]$$

where $k_q, i_q, j_q$ respectively denote the state, action of player 1 and action of player 2 at stage $q$.

The game previously described will denoted $\Gamma_n(p)$. $\Gamma_n(p)$ is a zero-sum game with $\Sigma$ and $\mathcal{T}$ as strategies spaces and payoff function $\gamma_n^p$. Furthermore, a standard argument implies that this game has a value, denoted $V_n(p)$, and players have optimal strategies.

## 5.3 Recursive formula

For each probability $p \in \Delta(K)$, the payoff function satisfies the following equation : $\forall \sigma \in \Sigma$, $\forall \tau \in \mathcal{T}$,

$$\gamma_N^p(\sigma,\tau) = \sum_{k\in K} p^k \gamma_N^{\delta_k}(\sigma,\tau)$$

Now, we give the recursive formula for the value $V_n$. First, we introduce several classical notations. Consider that actions of player 1 at the first stage are chosen accordingly to $(x^k)_{k\in K} \in \Delta(I)^K$. The probability that player 1 plays at stage 1 an action $i$ in $I$ is :

$$x(i) = \sum_{k\in K} p^k x^k(i)$$

And similarly, for each $i$ in $I$, the conditional probability induced on stage of nature given that player 1 plays $i$ at stage 1 is denoted $p^1(i) \in \Delta(K)$. We get

$$p^1(i) = \left(\frac{p^k x^k(i)}{x(i)}\right)_{k\in K}$$

**Remark 5.3.1** *If $x(i)$ is equal to $0$, then $p^1(i)$ is chosen arbitrarily in $\Delta(K)$.*

If player 2 plays $y \in \Delta(J)$, the expected payoff for player 1 is then

$$G(p,x,y) = \sum_{k\in K} p^k G^k(x^k,y)$$

Now, we describe the recursive operators associated to this game : we have for all $p \in \Delta(K)$

$$\underline{T}_G^M(V)(p) := \max_{x\in\Delta(I)^K} \min_{y\in\Delta(J)} \left( G(p,x,y) + \sum_{i\in I} x(i) V(p^1(i)M) \right)$$

$$\overline{T}_G^M(V)(p) := \min_{y\in\Delta(J)} \max_{x\in\Delta(I)^K} \left( G(p,x,y) + \sum_{i\in I} x(i) V(p^1(i)M) \right)$$

The following theorem, corresponding to proposition 5.1 in [1], gives the recursive formula linking $V_n$ and $V_{n-1}$.

**Proposition 5.3.2** *For all $n \geq 1$ and $p \in \Delta(K)$,*

$$V_n(p) = \underline{T}_G^M(V_{n-1})(p) = \overline{T}_G^M(V_{n-1})(p)$$

*In the following, we note $T_G^M = \overline{T}_G^M = \underline{T}_G^M$.*

The previous recursive formula is an essential tool to provide a recursive implementation of the value. Now, we are going to translate this recursive formula in order to reveal a parametric linear program, which will be able to be solved with an appropriate algorithm. First, we state the result we will prove in the next sections :

**Theorem 5.3.3** If $K = \{1, 2\}$ then for all $n \in \mathbb{N}$, $V_n$ is concave, piecewise linear. Furthermore, if $V_n$ is equal to $\min_{s\in[1,m]} < L^s, . >$ then for any $p \in [0, 1]$

$$V_{n+1}(p) = \min_{D(\hat{L})} (pu_1 - pu_2 + (1-p)v_1 - (1-p)v_2)$$

with $\hat{L} = ML$ and $D(\hat{L})$ *equals to*

$$\begin{cases} \forall i \in I & u_1 - u_2 & - & \sum_j z[j]a^1_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[1] & \geq & 0 \\ \forall i \in I & v_1 - v_2 & - & \sum_j z[j]a^2_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[2] & \geq & 0 \\ & \sum_j z[j] & = & 1 & & & & \\ \forall i \in I & \sum_{k\in[1,m]} y[k,i] & = & 1 & & & & \\ \textit{Variables} \geq 0 & & & & & & & \end{cases}$$

As suggested by the previous theorem, we link first the recursive operator to a parametric linear program.

## 5.4 From recursive operator to linear programming

As in the theorem hypotheses, our framework of analysis is subjected to some additional assumptions. We now assume, once for all, that the cardinal of $K$ is equal to 2, hence we denote $K = \{1, 2\}$. Under this assumption, $p$ may be considered as an element of the interval $[0, 1]$ and the recursive operator $T_G^M$ becomes : for any $p$ in $[0, 1]$,

$$T_G^M(V)(p) = \max_{(x^1,x^2)\in\Delta(I)^2} \min_{y\in\Delta(J)} [px^1G^1y + (1-p)x^2G^2y + \sum_{i=1}^{l} x(i)V(p^1(i)M)]$$

First, we present the recursive formula under a more appropriated form : The initial probability $p$ and $(x^k)_{k\in K} \in \Delta(I)^K$ generates a probability $\Pi$ on $\Delta(K \times I)$ such that $\Pi[k, i] = p^k x^k(i)$, for all $i$ in $I$ and all $k$ in $K$. Let us also denote for all $i \in I$, $\Pi[K, i] = \sum_k \Pi[k, i]$ the marginal distribution of $\Pi$ on $I$ and $\Pi[i]$ the vector $(\Pi^1[i], \Pi^2[i])$ in $\mathbb{R}^2$. These lead to the following recursive writing

$$T_G^M(V)(p) = \max_{\Pi\in\Delta^p} \left[ \min_{j\in J} \left( \sum_{k\in\{1,2\}} \sum_{i\in I} \Pi[k,i]G^k_{i,j} \right) + \sum_{i=1}^{l} \Pi[K,i]V\left(\frac{\Pi[i]}{\Pi[K,i]}M\right) \right]$$

where $\Delta^p := \{\Pi \in \Delta(K \times I) | \sum_i \Pi[k,i] = p^k\}$.

The main property making it possible to use linear programming techniques will be the piecewise linearity of the value function. First we then analyze the behavior of operator $T_G^M$ on concave, piecewise linear functions. Let us assume in the following that $V$ satisfies these assumptions. Hence, there exists $\{L^s | s \in [1,m]\}$ a finite subset of $\mathbb{R}^2$ such that for any $a \in \Delta(K)$

$$V(a) = \min_{s \in [1,m]} < L^s, a >$$

where $L^s = (L^s[1], L^s[2]) \in \mathbb{R}^2$.
So, the positivity of $\Pi[K,i]$ for any $i \in I$, leads to

$$T_G^M(V)(p) = \max_{\Pi \in \Delta^p} \left[ \min_{j \in J} \left( \sum_{k \in \{1,2\}} \sum_{i \in I} \Pi[k,i] G_{ij}^k \right) + \sum_{i=1}^{l} \min_{s \in [1,m]} < L^s, \Pi[i]M > \right]$$

Next, we write differently the previous problem in order to reveal a linear program, hence we get
$T_G^M(V)(p) = \max\left(a_1 - a_2 + \sum_{i \in I}(b_1^i - b_2^i)\right)$ under the constraints :

$$C(L,p) := \left\{ \begin{array}{lllll} \forall j \in I & & a_1 - a_2 & \leq & \sum_{i,k} \Pi^k[i] G_{ij}^k \\ \forall i \in I & \forall s \in [1,m] & b_1^i - b_2^i & \leq & < L^s, \Pi[i]M > \\ & & \sum_i \Pi^1[i] & = & p \\ & & \sum_i \Pi^2[i] & = & 1-p \\ \text{Variables } \geq 0 & & & & \end{array} \right.$$

Let us observe that $< L^s, \Pi[i]M > = < ML^s, \Pi[i] >$. Furthermore, for all $s \in [1,m]$, we denote by $\hat{L}^s$ the vector $ML^s \in \mathbb{R}^2$. The standard form of the previous program is then
$T_G^M(V)(p) = \max\left(a_1 - a_2 + \sum_{i \in I}(b_1^i - b_2^i)\right)$ under the constraints :

$$C(\hat{L},p) := \left\{ \begin{array}{llllllll} \forall j \in I & a_1 - a_2 & - & \sum_i \Pi^1[i] G_{ij}^1 & - & \sum_i \Pi^2[i] G_{ij}^2 & \leq & 0 \\ \forall i \in I, s \in [1,m] & b_1^i - b_2^i & - & \hat{L}^s[1]\Pi^1[i] & - & \hat{L}^s[2]\Pi^2[i] & \leq & 0 \\ & \sum_i \Pi^1[i] & \leq & p & & & & \\ & -\sum_i \Pi^1[i] & \leq & -p & & & & \\ & \sum_i \Pi^2[i] & \leq & 1-p & & & & \\ & -\sum_i \Pi^2[i] & \leq & p-1 & & & & \\ \text{Variables } \geq 0 & & & & & & & \end{array} \right.$$

Finally, in order to obtain a parametric problem, we transform the previous linear program into its dual, in the sense of linear programming. Hence, we obtain

$T_G^M(V)(p) = \min\,(pu_1 - pu_2 + (1-p)v_1 - (1-p)v_2)$ under the constraints :

$$D(\hat{L}) := \begin{cases} \forall i \in I & u_1 - u_2 & - & \sum_j z[j]a^1_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[1] & \geq & 0 \\ \forall i \in I & v_1 - v_2 & - & \sum_j z[j]a^2_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[2] & \geq & 0 \\ & \sum_i z[j] & \geq & 1 \\ & -\sum_j z[j] & \geq & -1 \\ \forall i \in I & \sum_{k\in[1,m]} y[k,i] & \geq & 1 \\ \forall i \in I & -\sum_{k\in[1,m]} y[k,i] & \leq & -1 \\ \\ & \text{Variables} & \geq & 0 \end{cases}$$

And the standard form of the previous problem becomes
$T_G^M(V)(p) = \min\,(pu_1 - pu_2 + (1-p)v_1 - (1-p)v_2)$ under the constraints :

$$D(\hat{L}) = \begin{cases} \forall i \in I & u_1 - u_2 & - & \sum_j z[j]a^1_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[1] & \geq & 0 \\ \forall i \in I & v_1 - v_2 & - & \sum_j z[j]a^2_{ij} & - & \sum_{k\in[1,m]} y[k,i]\hat{L}^k[2] & \geq & 0 \\ & \sum_j z[j] & = & 1 \\ \forall i \in I & \sum_{k\in[1,m]} y[k,i] & = & 1 \\ \\ & \text{Variables} & \geq & 0 \end{cases}$$

So, the value analysis is straightforward related to the analysis of a parametric linear program. In the following section, we will give an algorithmic resolution method for a general parametric linear program. And proposition 5.3.2 will allow us to compute recursively the value of the repeated game.

## 5.5 Parametric linear programming

Let us consider in the following, the parametric problem

$$(S_p) = \begin{cases} \min(c(p)x) \\ Ax = b \\ x \geq 0 \end{cases}$$

where $A$ is a matrix with $m$ rows, $n$ columns $(m \leq n)$, $b$ a $m$-vector column column, $c(p) := (e + pf)$ called vector cost, with $e$ a $n$-vector row, $p$ a scalar in $[0,1]$ and $f$ a $n$-vector row. We observe immediately that

**Remark 5.5.1** *The set of feasible solution of $(S_p)$ does not depend on the parameter $p$.*

Furthermore, we make the additional assumption : $D = \{x/Ax = b, x \geq 0\}$ is non empty. This hypothesis will allow us in particular to initialize the solving algorithm described below. In the following, we note $z(p)$ the value of objective function at optimum, of the problem $(S_p)$.

### 5.5.1 Heuristic approach

We may write $(S_p)$, for a point $p = p_0$, under its canonical form associated to an optimal basis. Heuristically, as in remark 5.5.1, we infer that there exists a neighborhood of $p_0$ for which the basis is always optimal. Hence, we may browse interval $[0, 1]$ and provide intervals having an unchanged optimal basis. In this way, given that we may compute the function $z$ for each extreme points of previous intervals, we are able to provide explicitly the function $z$.

In the following paragraph, we are going to describe a practical resolution method allowing to exhibit these intervals and we will also prove that there are a finite number of such intervals covering $[0, 1]$.

First, we give the heuristic way of analysis for a linear parametric program. We start with a value of $p$, $p = p_0$, and we are determining the proceeding to browse interval $[0, 1]$. The main tool of this analysis is the following step :

Let $p := p_0$ for which $(S_{p_0})$ possesses an optimal solution. We write $(S_{p_0})$ under its canonical form in relation to the optimal basis $J$ for $p = p_0$. If we keep the literal form of the objective function, the corresponding reduced costs depend naturally on $p$. More precisely, the reduced costs are linear in $p$. Let us denote $\hat{J}$ the complementary of $J$, and $(c_j(p))_{j\in\hat{J}}$ the reduced costs associated to the canonical writing. Since $J$ is optimal, we already know that $c_j(p_0) \geq 0$. In order to determine the set of points $p \geq p_0$ for which $J$ stays optimal for $(S_p)$, we analyze the dependency on $p$ of the reduced costs. It then appears two cases :

$(a)_{p_0}$ For all $j$ in $\hat{J}$ such that $c_j(p_0) = 0$, the coefficient of $p$ in $c_j(p)$ is $\geq 0$.

$(b)_{p_0}$ $\exists j_0 \in \hat{J}$ such that $c_{j_0}(p_0) = 0$, and coefficient of $p$ in $c_{j_0}(p)$ is $< 0$.

In case $(a)_{p_0}$, given that the reduced costs are linear in $p$, there exists $p_1 > p_0$ such that $J$ stays optimal on interval $[p_0, p_1]$.
In case $(b)_{p_0}$, the set of $p \geq p_0$ for which $J$ stays optimal is reduced to the singleton $\{p_0\}$. Finally, in order to provide a range of value for which basis $J$ stays optimal, we have to find an optimal basis verifying the constraint $(a)_{p_0}$. In the following section, we will determine the proceeding allowing to find such a basis. For the moment, we admit that we can provide one.
In the following, we will call "*main step*" the proceeding which gives a optimal basis verifying $(a)_{p_0}$.

The *"main step"* allows us to describe explicitly the parametric linear program value. For this, we have to use again the *"main step"* from $p = p_1$. And so, we get a point $p_2 > p_1$ and a basis staying optimal on $[p_1, p_2]$. In this way, we determine a sequence of points $(p_i)$ verifying $p_{i+1} > p_i$, $p_i$ will correspond to vertices abscises of function $z$. And the process stops when $p_i = 1$.

In order to prove the convergence of our method, we have in particular to show that the *"main step"* is a convergent algorithm and that it will be used a finite number of times. The next section is devoted to the elaboration of this algorithm.

### 5.5.2 Algorithm for $(S_p)$.

This section is split in three parts : firstly, we introduce another useful problem for which the notion of optimal basis verifying $(a_{p_0})$ appears naturally, secondly we focus our analysis on the convergence of algorithm giving such a basis, and finally we provide the entire method to express explicitly function $z$.
First, we define an order relation $\preceq$ on the set $\mathcal{P}$ of polynomial function of degree equal 1.

**Definition 5.5.2** *Let $P$ and $Q$ be in $\mathcal{P}$ and $a$ in $[0,1]$,*

1. *$P$ is negative : $P \preceq_a 0$ if there exists $h > 0$, such that $P$ is negative on interval $[a, a+h]$.*
2. *$P$ is strictly negative : $P \prec_a 0$ if there exists $h > 0$, such that $P$ is strictly negative on $]a, a+h]$.*
3. *$P \preceq_a Q$ (resp. $P \prec_a Q$) if $P - Q \preceq_a 0$ (resp. $P - Q \prec_a 0$).*

These definitions lead us to the following classical properties

**Proposition 5.5.3**

1. *For all $a$ in $[0,1]$, the relation $\preceq_a$ is a total order on $\mathcal{P}$.*
   *Let $P$ and $Q$ be in $\mathcal{P}$ :*
2. *If $P \preceq_a 0$ then $P(a) \leq 0$.*
3. *If $P \prec_a 0$ then $P(a) \leq 0$.*
4. *If $P$ is not $\preceq_a$ than $0$ then $0 \prec_a P$.*
5. *If $P \preceq_a 0$ and $Q \prec_a 0$ then $P + Q \prec_a 0$.*
6. *If $P + Q \prec_a 0$ then $P \prec_a 0$ or $Q \prec_a 0$.*
7. *If $c \in \mathbb{R}^{+,*}$ and $P \prec_a 0$ then $cP \prec_a 0$.*

**Remark 5.5.4** *Let $J$ a feasible basis for $(S_{p_0})$, let us observe that associated reduced costs $(c_j(p))_{j \notin J}$ are in $\mathcal{P}$.*
*Furthermore, if for all $j \notin J$, $0 \preceq_{p_0} c_j$ then :*

1. *$J$ is an optimal basis for $(S_{p_0})$.*
2. *$J$ verifies $(a)_{p_0}$.*

Thus, the previous remark leads us to the definition :

**Definition 5.5.5** *A basis $J$ is said to be $\preceq_{p_0}$-optimal if $J$ is optimal for the minimization problem $(S_{p_0})$ for the order $\preceq_{p_0}$ : this new problem will be denoted $(S_{p_0}^{\preceq})$.*

Next, we may connect the previous definition to our problematic

**Proposition 5.5.6** *$B$ is an optimal basis of $(S_{p_0}^{\preceq})$ if and only if $B$ is an optimal basis of $(S_{p_0})$ verifying $(a)_{p_0}$.*

So, It remains to prove the existence of such a basis and also to give a convergent algorithm which provides it. In this way, we first analyze the problem $(S_{p_0}^{\preceq})$ and we connect problem $(S_{p_0}^{\preceq})$ to initial problem $(S_{p_0})$, in particular : is there a link between optimal basis solutions ?
Let us denote $z_{p_0}^{\preceq}$ the value of minimization problem $(S_{p_0}^{\preceq})$, so point (2) in prop. 5.5.3 allows us to state

**Proposition 5.5.7** *For all $p_0$ in $[0,1]$,*

- *If $x_{p_0}^*$ is a basis $\preceq_{p_0}$-optimal solution of $(S_{p_0}^{\preceq})$ then $x_{p_0}^*$ is a basis optimal solution of $(S_{p_0})$.*
- *If $(S_{p_0})$ has an optimal solution then $(S_{p_0}^{\preceq})$ has a $\preceq_{p_0}$-optimal solution and*

$$z_{p_0}^{\preceq}(p_0) = z(p_0)$$

**Remark 5.5.8** *On the other hand, we remark that a basis optimal solution of $(S_{p_0})$ isn't necessarily a basis $\preceq_{p_0}$-optimal solution of $(S_{p_0}^{\preceq})$.*

Hence, we now focus our analysis on the problem $(S_{p_0}^{\preceq})$. And we give the proceeding which provide an $\preceq_{p_0}$-optimal basis.

This proceeding occurs in three steps :

1. Initialization
2. Iteration
3. End of the process

We focus our analysis on the two last steps. Initialization step is just a linear algebra exercise : Find a feasible basis.

**Iteration** step :

In this paragraph, we will introduced a improved release of Simplex algorithm. Iteration method is similar, we will simply use order $\prec_{p_0}$ on reduced costs to determine entering variables. A precise description of Simplex algorithm may be found in [3]. General proceeding is the following :

Initialization step provide us a feasible basis, assumed not $\preceq_{p_0}$-optimal. Our first goal is to determine an *entering* variable (non basis variables becoming basis), permitting to decrease, according to order $\preceq_{p_0}$, the value function we have to minimize. This "entering" variable determine a *leaving* variable. We get in this way a new basis, furthermore the objective function evaluated at the associated basis solution is less than the value obtained with the previous basis. In other words

Entering variable choice :

The variables which are candidates to be entering are non-basis variables having a reduced cost $c_j \prec_{p_0} 0$ in objective function. It probably exists several candidates. For the moment, choice will be not considered. We will see in the step " End of process " that this choice plays a central role. If no candidate exists, we have thus an $\preceq_{p_0}$-optimal basis.

$$\text{Entering variable } i : i \text{ such that } c_i \prec_{p_0} 0.$$

Leaving variable choice :

The leaving variable is a basis variable. According to the canonical expression, we may write basis variables in function of non-basis one. We choose as leaving variable, the first variable becoming non basis, which means becoming null when the value of the entering variable increases. If there is several candidates, the choice will be considered in the following.

$$\text{Leaving variable} j : j \text{ solution of } \min_{\{j|A_{i,j}>0\}} \frac{b_j}{A_{i,j}}.$$

Previous proceeding gives a new feasible basis for $(S_{p_0}^{\preceq})$, if this basis is $\preceq_{p_0}$-optimal, the process stops. In the contrary case, we iterate the proceeding as long as a $\preceq_{p_0}$-optimal basis doesn't appear.
This method raises the following question : Does the process stop ? The choice

of entering and leaving variables may generate the same system in two different iterations of the problem. In this case, the process is said to cycle. So, Does algorithm cycle ?

Now, we focus our analysis on the **end of the process**.
First, we state a classical result concerning the Simplex method, this result also works in our framework :

**Proposition 5.5.9** *If process does not stop then it cycles.*

In this case, a simple rule permits to delete the cycle possibilities. This rule, in the simplex case, is due to Robert Bland. Firstly, we arbitrarily associate a number, called index, to each variable of our problem. In the case, where several variables are candidates to enter or to leave the basis, we choose the variable which has the smallest index. The choice is the following :

1. Entering variable $i$ : minimum $i$ such that $c_i \prec_{p_0} 0$.
2. Leaving variable $j$ : minimum $j$ such that $j$ solution of $\min_{\{j|A_{i,j}>0\}} \frac{b_j}{A_{i,j}}$.

Hence, the following proposition guarantees the convergence of our method

**Proposition 5.5.10** *If the entering variable choice is made accordingly to the Bland rule, the process does not cycle.*

**Proof :** The proof is similar to the classic one, we have just to use $\prec_{p_0}$ instead of $<$.□

Finally, for all $p_0$, we obtain a converging algorithm giving a $\preceq_{p_0}$-optimal basis. Hence, a crucial point in the determination of $z$ is the following proposition, which is a reformulation in our case of the fundamental theorem of linear programming.

**Proposition 5.5.11** *If $(S^{\preceq}_{p_0})$ has an $\preceq_{p_0}$-optimal solution then it has a basis $\preceq_{p_0}$-optimal solution.*

So, according to the description made in the heuristic approach and proposition 5.5.7, at each $p_0$ we are able now to provide an optimal basis verifying $(a_{p_0})$. Hence, to conclude the convergence of the entire method, it is sufficient to answer to the previously asked question : In order to cover $[0, 1]$, does a finite number of iterations of the "main step" appear ?

**Remark 5.5.12** *Indeed, algorithm may reproduce an infinite number of the "main step" on one piece of linearity of $z$.*

To answer this question, we have just to observe the following facts :
– *"The subset of* $[0,1]$ *on which a basis stays optimal is an interval."*
– *" The number of basis is finite. "*
The first fact is obviously true, it is simply due to the fact that the reduced costs are in $\mathcal{P}$.

So, the proceeding to browse interval $[0,1]$ is the following :

If we start the resolution process from $p_0 = 0$, with the main step, we then find an $\preceq_0$-optimal basis $B_0$ and a point $p_1 > p_0$ such that $B_0$ stays optimal on interval $[p_0, p_1]$. We may also assume that $p_1$ is maximal for this property. Next, applying the main step to the point $p_1$, we thus find a $\preceq_{p_1}$-optimal basis $B_1$ and a point $p_2 > p_1$ such that $B_1$ stays optimal on interval $[p_1, p_2]$. By the maximality property of $p_1$, $B_1$ is obviously different of $B_0$. If we recursively apply this proceeding, we then obtain an increasing sequence of points $(p_i)_i$ in $[0,1]$ and a sequence of basis $B_i$ such that :
– $B_i$ is optimal on $[p_i, p_{i+1}]$.
– $p_{i+1}$ is the greater point such that $B_i$ verifies the previous constraint.
Let us observe, by the maximality property of points $p_i$, that $B_i$ and $B_{i+1}$ are distinct. Furthermore, since the set of points for which $B_{i+1}$ stays optimal is an interval, so, $B_{i+1}$ and $B_k$ for $k \leq i$ are thus different. Then, since the problem has a finite number of basis, we then deduce that : there exists $i_0$ such that $p_{i_0} = 1$.

Finally, our algorithm is thus convergent and we get the following theorem

**Theorem 5.5.13** *$z$ is concave, piecewise linear on* $[0,1]$.
*Furthermore, There exists a finite set of points $(p_i)_{i=0,\ldots,s}$ in $[0,1]$ with $p_0 = 0$ and $p_s = 1$ and finite set of basis $(J_i)_{i=0,\ldots,s-1}$, such that for all $i = 0,\ldots,s-1$, $J_i$ is optimal on $[p_i, p_{i+1}]$.*

**Remark 5.5.14 (Algorithm Complexity)**
*In this kind of proceeding, It is very difficult to provide precisely the complexity. We do not have any information on the number of "main step" effectuated, we only know that this number is bounded by the cardinal of the set of basis, which is itself bounded by $C_n^m$. Finally, we only know that complexity is bounded by $S(m,n)C_n^m$, with $S(m,n)$ the simplex complexity for a $m \times n$-matrix $A$. Since we apply this process recursively this kind of complexity computation generates an accumulation of errors. This analysis is very vague, and we have no further information concerning exact complexity of our algorithm.*

# 5.6 Induced results

As a direct consequence of previous results, we get

**Theorem 5.6.1**
*If $V$ is concave piecewise linear of the form $\min_{s\in[1,m]} < L^s, . >$ then $T_G^M(V)$ is concave piecewise linear. Furthermore, for all $p \in [0,1]$*

$$T_G^M(V)(p) = \min_{D(\hat{L})} (pu_1 - pu_2 + (1-p)v_1 - (1-p)v_2)$$

*with $\hat{L} = ML$.*

And theorem 5.3.3 is then proved as an obvious corollary. In the following section, we provide semi-code allowing to implement algorithm which computes $V_n$.

## 5.6.1 Algorithm for the repeated game value

In this section, we provide the code giving the entering variable and the "main step", the others proceeding may be computed in a similar way as the simplex algorithm. Now, let us assume that the linear program is written under the canonical form associated to a basis $B$. So, the function we have to minimize may be written as $f(p,x) := \alpha(p) + \sum_{j\notin B} c_i(p), x_j$, with $\alpha$ and $c_j$ in $\mathcal{P}$.

Choice of entering variable

| |
|---|
| **Input :** The function $f$ and $p_0$ in $[0,1]$<br>**Output :** Entering variable $y$ if it exists, $Fail$ otherwise. |
| Let $F_0$ be the empty set.<br>**For** $j$ not in $B$ **do** :<br>**If** $c_j(p_0) < 0$ **then** $F_0 := F_0 \cup \{x_j\}$ **EndIf** :<br>**If** $c_j(p_0) = 0$ and coefficient of $p$ in $c_j$ is $< 0$ **then** $F_0 := F_0 \cup \{x_j\}$ **EndIf** :<br>**Enddo** :<br>**If** $F_0 \neq$ emptyset **then**<br>$y := x_j$, with $j$ minimum such that $x_j \in F_0$.<br>**Else** $y := Fail$ :<br>**EndIf** :<br>Exit $y$ : |

Furthermore, let us assume that $B$ is $\preceq_{p_0}$-optimal, we keep the same writing for the function $f$. The following proceeding allows to determine the interval on which $B$ stays optimal.

<u>Interval on which $B$ stays optimal.</u>

| |
|---|
| **Input :** The reduced costs $c_j$ for $j \notin B$.<br>**Output :** Point $p_1$ such that $B$ is optimal on $[p_0, p_1]$,<br>and maximal for this property. |
| Let $P_0$ be the empty set.<br>**For** $j$ not in $B$ **do** :<br>**If** coefficient of $p$ in $c_j$ is $< 0$ **then** $P_0 := P_0 \cup \{\text{solution of } c_j(p) = 0\}$ **EndIf** :<br>**Enddo** :<br>$p_1 := min_{a \in P_0}(a)$ :<br>Exit $p_1$ : |

The two previous steps allows us to compute explicitly the function $z$, with its intervals of linearity. Finally, we are able now to solve the problem stated in theorem 5.6.1. In the following, we will name "ProgParam$_G^M$ " the proceeding which takes as input : A concave piecewise linear function $V := \min_{s \in [1,m]} < L^s, . >$ and which gives as output : the function $T_G^M(V)$ corresponding to the parametric linear program given in theorem 5.6.1. In other words,

<u>"ProgParam$_G^M$ "</u>

| |
|---|
| **Input :** A finite set of points in $\mathbb{R}^2$ : $(L^s)_{s \in [1,m]}$<br>(corresponding to $V := \min_{s \in [1,m]} < L^s, . >$) |
| **Output :** A finite set of points in $\mathbb{R}^2$ : $(\tilde{L}^s)$<br>(corresponding to $T_G^M(V) := \min_s < \tilde{L}^s, . >$) |

Now, we may provide the recursive proceeding computing $V_n$ starting from $V_0 = \min_{s \in [1,m]} < L_0^s, . >$.

So, we now implement recursively the process and we will denote $V(n, L_0, G^1, G^2, M)$ the following algorithm, which gives explicitly the value $V_n$ and also the running time. This function will permit us to know if $V_n$ reaches a fixed point of the recursive operator and also the first step for which this happens.

$\underline{V(n, L_0, G^1, G^2, M)}$

**Input :** $n$ : The length of the game.
$L_0$ : A finite set of points in $\mathbb{R}^2$. (Corresponding to $V_0$)
$G^1$ and $G^2$ : payoff matrices of the game.
$M$ : The transition matrix of the Markov chain.
**Output :** - All values $V_i$, $i$ between 1 and $n$, under the form of a finite number of points in $\mathbb{R}^2$ : $L_i := (L_i^s)$ such that $V_i := \min_{s\in[1,m]} < L_i^s, . >$.
- $t$ : Running time.
- $d$ : Number of iteration without reaching a fixed point.

Let $t_0$ := time at the beginning.
$L$ := a sequence of points such that $L(0) := L_0$ :
$d := 0$ :
**For** $i$ from 1 to $n$ **do** :
$L(i)$ :=ProgParam$_G^M$ $(L(i-1))$
$d := i$ :
**If** $L(i) = L(i-1)$ then $i := n$ **EndIf** :
**Enddo** :
$t_1$ := time at the end.
$t := t_1 - t_0$.
Exit : $(L(i))_{i=1\dots,d}$, $t$, $d$.

Finally, this proceeding allows us to draw and to visualize graphically the values $V_1, \dots, V_n$. In the next section, we now apply this algorithm to several known examples.

## 5.7 Examples

### 5.7.1 A particular Markov chain game

In this section, we deal with an example introduced in [1], and we give a partial answer to the question addressed by the author. Furthermore, we provide some graphs which allow to get intuition concerning the repeated game values.
Let us first define the transition matrix $H$ of the game : $H := \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$
And the payoff matrices of player 1,

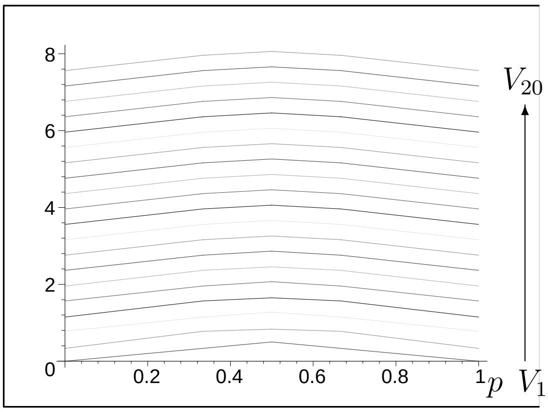
$$G^1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, G^2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We give two results , each of them associated to a different number of iterations $n$ : $n = 20$, $n = 60$. We remind that

- $n$ corresponds the length of the game.
- "End" corresponds to the number of effectuated steps before reaching a fixed point. In other words, if "End"$=j < n$ then $V_j = V_{j+k}$ for all $k \in \mathbb{N}$.
- "running time" corresponds to the running time of my computer in seconds.

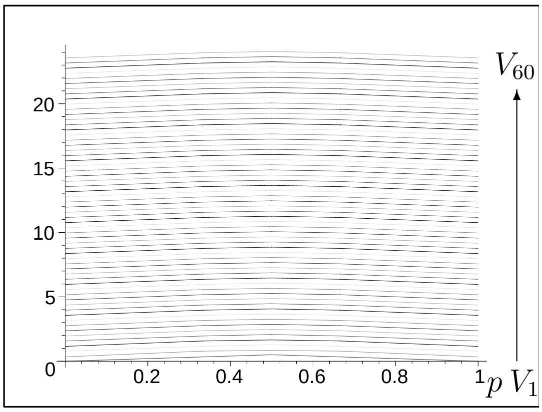
In the following graphs, we draw the functions $V_n$ and abscise corresponds to $p \in [0, 1]$.

**<u>n = 20</u>**



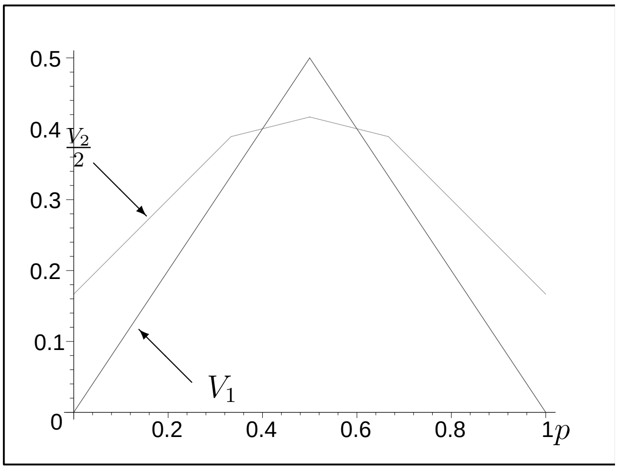

"End" $=$ 20
"running time" $=$ 9.324

**<u>n = 60</u>**




"End" $=$ 60
"running time" $=$ 31.305

Furthermore, the following graph answers precisely to the question addressed by J. Renault in [1] : "*The value* $\frac{V_n}{n}$ *is not decreasing*". Indeed, author show that $V_1(\delta_1) = 0 < \frac{V_2}{2}(\delta_1) = \frac{1}{6}$, et he concludes that the value is not decreasing. But concerning this example, he gives no further information, for example : Is it increasing ? . The following graph confirms this results and show that the sign of $V_1 - \frac{V_2}{2}$ changes on $[0, 1]$.




Now, the last examples deal with classical repeated games with lack of information on one side, which means that matrix $H$ is equal to the identity matrix.

### 5.7.2 Explicit values : Mertens Zamir example

We consider the following two state game :

$$G^1 := \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}, G^2 := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Let us define $b(k,n) = \binom{n}{k}2^{-n}$, $B(k,n) = \sum_{m\le k} b(m,n)$, for $0 \le k \le n$ and $B(-1,n) = 0$. Let also $p_{k,n} = B(k-1,n)$, $k = 1,\ldots,n+1$, Heuer in [4] has proved that $V_n$ is linear on each interval $[p_{k,n}, p_{k+1,n}]$ with value $V_n(p_{k,n}) = \frac{n}{2}b(k-1,n-1)$. With our proceeding, we get the following values $V_n$ : they are given under the form

$$\text{“}V\text{“}(n) = [[p_{0,n}, V_n(p_{0,n})], \ldots, [p_{k,n}, V_n(p_{k,n})], \ldots, [p_{n,n}, V_n(p_{n,n})]]$$

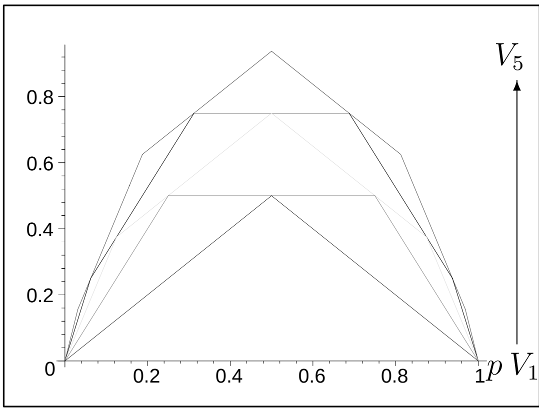
So, we obtain for $n = 1, 2, 3$ :

$$\text{“V”}(1) = [[0,\ 0],\ [\frac{1}{2},\ \frac{1}{2}],\ [1,\ 0]]$$

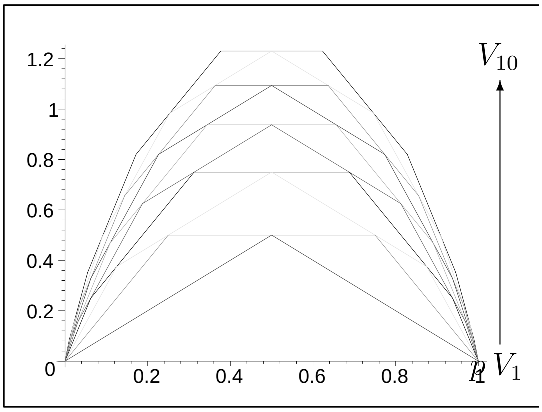
$$\text{“V”}(2) = [[0,\ 0],\ [\frac{1}{4},\ \frac{1}{2}],\ [\frac{1}{2},\ \frac{1}{2}],\ [\frac{3}{4},\ \frac{1}{2}],\ [1,\ 0]]$$

$$\text{“V”}(3) = [[0,\ 0],\ [\frac{1}{8},\ \frac{3}{8}],\ [\frac{1}{4},\ \frac{1}{2}],\ [\frac{1}{2},\ \frac{3}{4}],\ [\frac{3}{4},\ \frac{1}{2}],\ [\frac{7}{8},\ \frac{3}{8}],\ [1,\ 0]]$$

Finally, we may easily verify that we obtain the same values.
And the corresponding graphs are




"End" $= 5$
"running time" $= 4.156$



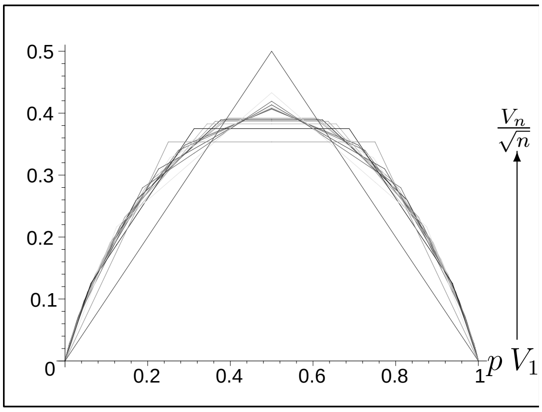

"End" $= 10$
"running time" $= 16.453$

### 5.7.3 Convergence of $V_n/\sqrt{n}$ : Mertens Zamir example

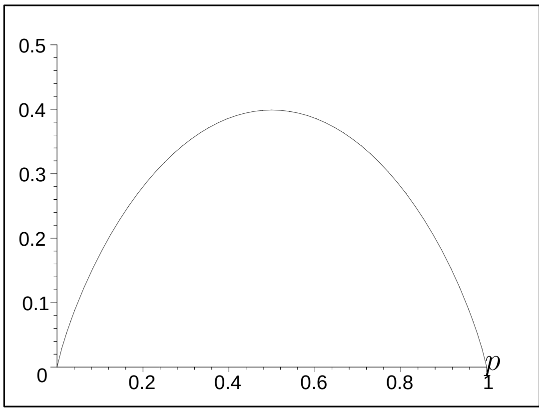
Furthermore, in this case Mertens and Zamir in [5] have proved that $V_n/\sqrt{n}$ converges to $\psi$ where $\psi(p)$ is the normal density function evaluated at its $p$-quantile. Which means that :

$$\psi(p) := \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_p)^2}{2}}, \text{ where } \frac{1}{\sqrt{2\pi}} \int_{x_p}^{+\infty} e^{-\frac{y^2}{2}} dy = p$$

On the two following graphs, we draw the sequence $\frac{V_n}{\sqrt{n}}$ for $n = 1, \ldots, 15$ and on the second one the graph of the function $\psi$.




"End" $= 15$




Graph of $\psi$

As we may see on the previous graphs, asymptotic behavior of the value appears quite naturally.

### 5.7.4 Fixed point : Market game example

In [2], De Meyer and Marino provide a fixed point of the recursive operator for a particular mechanism of exchange. In this paper, players have $l$ available actions and the payoff matrices are : for $i, j \in \{0, \ldots, l-1\}$, $l \in \mathbb{N}^*$.

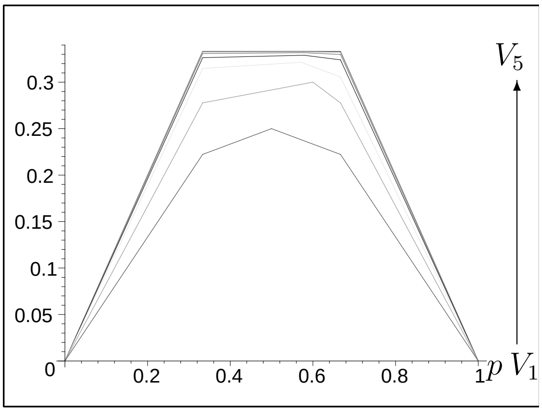
$$G^k_{ij} := 1\!\!1_{i>j}(1\!\!1_{k=1} - \frac{i}{l-1}) + 1\!\!1_{j>i}(\frac{j}{l-1} - 1\!\!1_{k=1})$$

For example, In the case $l = 4$, payoff matrices are the following

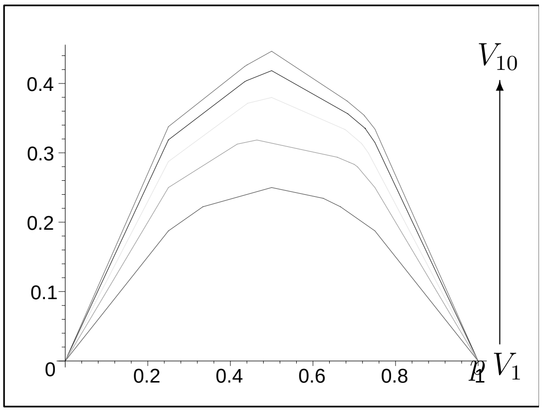
$$G^1 := \begin{pmatrix} 0 & \frac{-2}{3} & \frac{-1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{-1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, G^2 := \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ \frac{-1}{3} & 0 & \frac{2}{3} & 1 \\ \frac{-2}{3} & \frac{-2}{3} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

If players have $l$ actions, the recursive operator has a fixed point, noted $g^l$. $g^l$ being piecewise linear and piece of linearity corresponds to intervals $[\frac{i}{l-1}, \frac{i+1}{l-1}]$ for $i$ between 0 and $l-1$. Furthermore, for all $i$ such that $\frac{i}{l-1} \leq \frac{1}{2}$, we have $g^l(\frac{i}{l-1}) = \frac{li}{2(l-1)}$. In order to verify that $g^l$ is a fixed point of the recursive operator we first draw values $V_n$ for the game with $l = 4$ and $l = 5$.

$\underline{l{=}4}$ $\underline{l{=}5}$

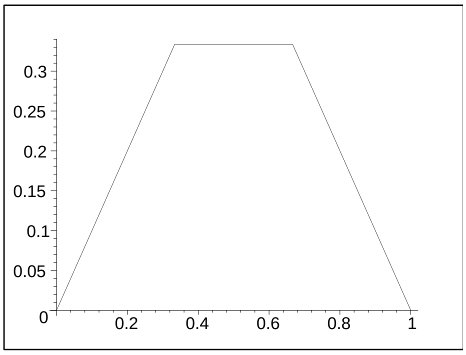



"End" $=5$
"running time" $=4.156$




"End" $=5$
"running time" $=16.453$

Furthermore, our program allows us to verify that $g^l$ is really a fixed point of the recursive operator. For example, in case $l=4$, the following graph corresponds to $V(10, g^4, G^1, G^2, Id)$,




"End" $=1$

Let us observe that the number of iteration is equal to 1, hence we deduce that $g^l$ is fixed point of the recursive operator.

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# Chapitre 6

# The value of a particular Markov chain game

A. Marino

In this paper, we give an explicit formula for the value of a particular Markov chain game. This kind of game was introduced in [1] by J.Renault. In that paper, the author analyzes a repeated zero-sum game depending essentially on the payoff matrices and on a Markov chain given by its transition matrix. The author provides a particular case with two states of nature for which he does not succeed to provide the value of infinite game. In this paper, we answer this question by determining the explicit formula of the value of finitely repeated game, which directly allows to provide the value of infinite game.

## 6.1 The model

This paper is split in two main parts : the first section is devoted to the description of the model introduced by J.Renault in [1] and the second one gives the proofs of theorems providing the explicit values of finitely and infinitely repeated games.

First, we remind the model introduced by J.Renault in [1]. If $\mathcal{S}$ is a finite set, let us define $\Delta(\mathcal{S})$ the set of probabilities on $\mathcal{S}$. Let us also denote by $K := \{1, \ldots, |K|\}$ the set of states of nature, where $|K|$ denotes the cardinal of the set $K$, $I$ the actions set of player 1 and $J$ those of player 2.

In the following, $K, I, J$ are supposed to be finite. In the particular case analyzed here, we will make the following additional assumptions : The cardinal of $K$, $I$ and $J$ will be equal to 2. In the general description of the model, these hypotheses will be not considered. Now, we introduce a family of $|I| \times |J|$-payoff matrices for player 1 : $(G^k)_{k\in K}$, and a Markov chain on $K$ defined by an initial probability $p$ on $\Delta(K)$ and a transition matrix $M = (M_{kk'})_{(k,k')\in K\times K}$. All elements of $M$ are supposed to be non negative and for all $k \in K$ : $\Sigma_{k'} M_{kk'} = 1$. Moreover, an element $q$ in $\Delta(K)$ may be represented by a row vector $q = (q^1, \ldots, q^{|K|})$ with $q^k \geq 0$ for any $k$ and $\sum_{k\in K} q^k = 1$.

The Markov chain properties give in particular that, if $q$ is the law on the states of nature at some stage, the law at the next stage is then $qM$. We denote, for all $k \in K$, $\delta_k$ the Dirac measure on $k$.

The play of the zero-sum game proceeds in the following way :

- At the first stage, probability $p$ initially chooses a state $k_1$ and only player 1 is informed of $k_1$. Players 1 and 2 independently choose an action $i_1 \in I$ and $j_1 \in J$, respectively. The payoff of player 1 is then $G^{k_1}(i_1, j_1)$, and $(i_1, j_1)$ is publicly announced, and the game proceed to the next step.
- At stage $2 \leq q \leq n$, probability $\delta_{k_{q-1}}M$ chooses a state $k_q$, only player 1 is informed of this state. The players independently select an action in their own set of actions, $i_q$ and $j_q$ respectively. The stage payoff for player 1 is then $G^{k_q}(i_q, j_q)$, and $(i_q, j_q)$ is publicly announced, and the game proceed to the next stage.

Let us note that payoffs are not announced after each stage. Players are assumed to have perfect recall, and the whole description of the game is a public knowledge.

Now, we remind the notion of behavior strategy in this game for player 1. A behavior strategy for player 1 is a sequence $\sigma = (\sigma_q)_{1\leq q\leq n}$ where for all $n \geq 1$, $\sigma_q$ is a mapping from $(K\times I\times J)^{q-1}\times K$ to $\Delta(I)$. In other words, $\sigma_q$ generate a mixed strategy at stage $q$ depending on past and current states and past actions played. As we can see in the game description, states of nature are not available for player 2, so a behavior strategy for player 2 is a sequence $\tau = (\tau_q)_{1\leq q\leq n}$, where for all $q$, $\tau_q$ is defined as a mapping from the cartesian product $(I \times J)^{n-1}$ to $\Delta(J)$. In the following, we denote by $\Sigma$ and $\mathcal{T}$, respectively, the set of behavior strategies of player 1 and player 2. According to $p$, a strategy profile $(\sigma, \tau)$ induces naturally a probability on $(K \times I \times J)^n$, and we denote $\gamma_n^p$ the expected payoff for player 1 :

$$\gamma_n^p(\sigma, \tau) := E_{p,\sigma,\tau}[\sum_{q=1}^{N} G^{k_q}(i_q, j_q)]$$

where $k_q, i_q, j_q$ respectively denote the state, action of player 1 and action of player 2 at stage $q$.

The game previously described will be denoted $\Gamma_n(p)$. $\Gamma_n(p)$ is a zero-sum game with $\Sigma$ and $\mathcal{T}$ as strategies spaces and payoff function $\gamma_n^p$. Furthermore, a standard argument gives that this game has a value, denoted $V_n(p)$, and players have optimal strategies.

In this paper, we determine an explicit formula for the value a particular Markov chain game. We assume that the state of nature is $K := \{1, 2\}$, the payoff matrices of player 1 are $G^1$ and $G^2$ such that

$$G^1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, G^2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the transition matrix $M$ equal to

$$M := \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Let us first observe that a probability on states of nature will be assimilated to a number in the interval $[0, 1]$, which corresponds to the probability of state 1. In this case, the values are concave functions from $[0, 1]$ to $\mathbb{R}$ and verify

**Theorem 6.1.1** *For all $n$ in $\mathbb{N}$, $V_n$ is piecewise linear on $[0, 1]$ of vertices*

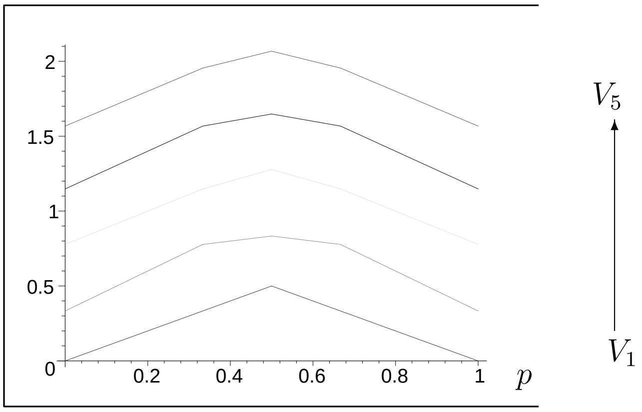
$$(0, \alpha_n), (\frac{1}{3}, \beta_n), (\frac{1}{2}, \gamma_n), (\frac{2}{3}, \beta_n), (1, \alpha_n)$$

*Furthermore, $\alpha_n$, $\beta_n$ and $\gamma_n$ verify the following recursive system*

$$\begin{cases} \alpha_{n+1} &= \beta_n \\ \beta_{n+1} &= \frac{1}{3}(1 + \beta_n + 2\gamma_n) \\ \gamma_{n+1} &= \frac{1}{2} + \beta_n \end{cases} \tag{6.1.1}$$

*with $\alpha_0 = \beta_0 = \gamma_0 = 0$*

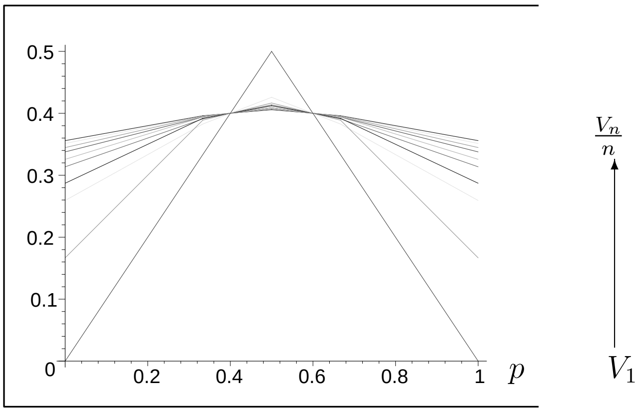
This result may be illustrated be the following graphs (see chapter 5) :




Furthermore, if we denote $\Gamma^{\infty}(p)$ the infinitely repeated game. J.Renault proved in [1] that this game has a value, denoted by $v_\infty$. Furthermore, we have $v_\infty = \lim_{n\to+\infty} \frac{V_n}{n}$. In particular, we obtain the desired result concerning the asymptotic behavior of the value.

**Corollary 6.1.2** *$v_\infty$ is equal to $\frac{2}{5}$.*

Similarly, this result may be view on the following graph :




This remaining parts of this paper will be split in two parts : the first section is devoted to the description of a very useful tool : The recursive formula linking $V_{n-1}$ to $V_n$, and the second one gives the proofs of theorem 6.1.1 and corollary 6.1.2.

## 6.2 Recursive formula

For each probability $p \in \Delta(K)$, the payoff function satisfies the following equation : $\forall \sigma \in \Sigma$, $\forall \tau \in \mathcal{T}$,

$$\gamma_N^p(\sigma, \tau) = \sum_{k \in K} p^k \gamma_N^{\delta_k}(\sigma, \tau)$$

We now give the recursive formula for the value $V_n$. We have first to introduce several classical notation. In the following, we take similar notations to those introduced in [1], for further information the reader will refer to this article. Consider that actions of player 1 at the first stage are chosen accordingly to $(x^k)_{k \in K} \in \Delta(I)^K$. The probability that player 1 plays at stage 1 an action $i$ in $I$ is :

$$\bar{x}(i) = \sum_{k \in K} p^k x^k(i)$$

And similarly, for each $i$ in $I$, the conditional probability induced on stage of nature given that player 1 plays $i$ at stage 1 is denoted $\bar{p}(i) \in \Delta(K)$. We get

$$\bar{p}(i) = \left( \frac{p^k x^k(i)}{\bar{x}(i)} \right)_{k \in K}$$

**Remark 6.2.1** *If $\bar{x}(i)$ is equal to* 0*, then $\bar{p}(i)$ is chosen arbitrarily in $\Delta(K)$.*

If player 2 plays $y \in \Delta(J)$, the expected payoff for player 1 is

$$G(p, x, y) = \sum_{k \in K} p^k G^k(x^k, y)$$

We can now describe the recursive operators associated to this game : for all $p \in \Delta(K)$

$$\underline{T}(V)(p) := \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)} \left( G(p, x, y) + \sum_{i \in I} \bar{x}(i) V(\bar{p}(i) M) \right)$$

$$\overline{T}(V)(p) := \min_{y \in \Delta(J)} \max_{x \in \Delta(I)^K} \left( G(p, x, y) + \sum_{i \in I} \bar{x}(i) V(\bar{p}(i) M) \right)$$

The following theorem, corresponding to proposition 5.1 in [1], gives the recursive formula for the value linking $V_n$ and $V_{n-1}$.

**Theorem 6.2.2** *For all $n \geq 1$ and $p \in \Delta(K)$,*

$$V_n(p) = \underline{T}(V_{n-1})(p) = \overline{T}(V_{n-1})(p)$$

*In the following, we denote $T$ the recursive operator.*

Furthermore, theorem 6.1 in [2] gives

**Theorem 6.2.3** *If $V$ is piecewise linear concave then $T(V)$ is concave piecewise linear.*

The previous recursive formula is an essential tool to provide an explicit formula for the value $V_n$. Now, we are going to analyze the particular case introduced above.

## 6.3 The particular case

We remind that in this particular a probability on states of nature will be assimilated to a number in the interval $[0,1]$, which corresponds to the probability of state 1. In particular, $\bar{p}(i)M$ is associated to the probability $\frac{\bar{p}^1(i)}{3} + \frac{1}{3}$, and without ambiguity, we will denote it : $\frac{\bar{p}(i)}{3} + \frac{1}{3}$.
Let us denote the sets of actions $I := \{H, B\}$ and $J := \{G, D\}$. So, in this case, operator $T$ becomes

$$T(V)(p) := \max_{x^1,x^2 \in \Delta(\{H,B\})} \min\left(\bar{x}(H)\bar{p}(H), \bar{x}(B)(1-\bar{p}(B))\right) + \sum_{i\in\{H,B\}} \bar{x}(i)V(\frac{\bar{p}(i)}{3}+\frac{1}{3})$$

Since $\bar{x}(H)\bar{p}(H) + \bar{x}(B)\bar{p}(B) = p$, and $\bar{x}(H) = 1 - \bar{x}(B)$, we get

$$\min\left(\bar{x}(H)\bar{p}(H), \bar{x}(B)(1-\bar{p}(B))\right) = \bar{x}(H)\bar{p}(H) + \min\left(0, 1-p-\bar{x}(H)\right)$$

And so,

$$T(V)(p) := \max_{x^1,x^2 \in \Delta(\{H,B\})} \bar{x}(H)\bar{p}(H) + \min\left(0, 1-p-\bar{x}(H)\right) + \sum_{i\in\{H,B\}} \bar{x}(i)V(\frac{\bar{p}(i)}{3}+\frac{1}{3}) \tag{6.3.1}$$

For a lot of clarity, it is useful to use another parametrization of player 1 strategy space : The space of pair $(\bar{x}, \bar{p})$ such that $\bar{x} \in \Delta(\{H,B\}) = [0,1]$, $\bar{p} : \{H,B\} \to [0,1]$ such that $\bar{x}(H)\bar{p}(H) + \bar{x}(B)\bar{p}(B) = p$ may be identified with the space of $(\sigma_1, \sigma, P)$, with $P : [0,1] \to [0,1]$, $\sigma \in [0,1]$ and $\sigma_1 \in [0, 1-\sigma]$ satisfying :

$$\left\{ \begin{array}{ll} (1) & \int_0^1 P(u)du = p \\ (2) & P \text{ is constant on each sets } [\sigma_1, \sigma_1+\sigma] \text{ and } [0,1]\backslash[\sigma_1, \sigma_1+\sigma]. \end{array} \right. \tag{6.3.2}$$

Given such a element $(\sigma_1, \sigma, P)$, player 1 plays as follows : $\bar{x}(H)$ corresponds to $\sigma$ and $\bar{p}(H) = P(u)$ if $u \in [\sigma_1, \sigma_1+\sigma]$ and $\bar{p}(B) = P(u)$ if $u \in [0,1]\backslash[\sigma_1, \sigma_1+\sigma]$, in this case, we obtain $p = \int_0^1 P(u)du = \sigma\bar{p}(H) + (1-\sigma)\bar{p}(B) = \bar{x}(H)\bar{p}(H) + \bar{x}(B)\bar{p}(B)$. Conversely, any pair $(\bar{x}, \bar{p})$ may be obviously generated in this way. So, we may now view the maximization problem in (6.3.1) as a maximization over the set $(\sigma_1, \sigma, P)$ satisfying (6.3.2), then (6.3.1) becomes

$$T(V)(p) := \max_{(\sigma_1,\sigma,P)} \int_{\sigma_1}^{\sigma_1+\sigma} P(u)du + \min(0, 1-p-\sigma) + \int_0^1 V(\frac{P(u)}{3} + \frac{1}{3})du$$

Let us observe that $P$ can take almost two value, let us denote $p^+$ and $p^-$ these value with $p^+ \geq p^-$. If we fix $\sigma$, the optimal behavior for player 1 for $\sigma_1$ and $P$ in this recursive formula is then to fix $\sigma_1 = 0$ and $P$ such that $P = p^+$ on $[0,\sigma]$ and $P = p^-$ on $[\sigma, 1]$. The recursive formula becomes

$$T(V)(p) := \max_{0 \leq p^- \leq p^+ \leq 1, \sigma p^+ + (1-\sigma)p^- = p} \sigma p^+ + \min(0, 1-p-\sigma) + \int_0^1 V(\frac{P(u)}{3} + \frac{1}{3})du$$

Furthermore, the optimal action for player 1 is to fix $\sigma = 1-p$. Indeed, since $P$ is $[0,1]$-valued and $\int_0^1 P(u)du = p$, all another actions is dominated by $\sigma = 1-p$. Hence the recursive formula becomes

$$T(V)(p) := \max_{0 \leq p^- \leq p^+ \leq 1, (1-p)p^+ + pp^- = p} (1-p)p^+ + (1-p)V(\frac{p^+}{3} + \frac{1}{3}) + pV(\frac{p^-}{3} + \frac{1}{3})$$

Furthermore, we now assume that $V$ is piecewise linear with vertices

$$(0, \alpha_n), (\frac{1}{3}, \beta_n), (\frac{1}{2}, \gamma_n), (\frac{2}{3}, \beta_n), (1, \alpha_n)$$

In particular, $V(p) = V(1-p)$. First, let us observe that $T(V)$ is also symmetric. Indeed, if $(p^+, p^-)$ is optimal in the previous problem then, since $V$ is symmetric, $T(V)(p)$ is equal to

$$\begin{array}{rl} & (1-p)p^+ + (1-p)V(\frac{p^+}{3} + \frac{1}{3}) + pV(\frac{p^-}{3} + \frac{1}{3}) \\ = & p(1-p^-) + (1-p)V(1 - (\frac{p^+}{3} + \frac{1}{3})) + pV(1 - (\frac{p^-}{3} + \frac{1}{3})) \\ = & p(1-p^-) + (1-p)V(\frac{1-p^+}{3} + \frac{1}{3}) + pV(\frac{1-p^-}{3} + \frac{1}{3}) \end{array}$$

So, let us denote temporarily $q = 1-p$, $\tilde{p}^- = 1-p^+$, and $\tilde{p}^+ = 1-p^-$, so, we get $q\tilde{p}^- + (1-q)\tilde{p}^+ = q$ and so

$$\begin{aligned} &= (1-q)\tilde{p}^+ + qV(\tfrac{\tilde{p}^-}{3}+\tfrac{1}{3}) + (1-q)V(\tfrac{\tilde{p}^+}{3}+\tfrac{1}{3}) \\ &\leq T(V)(1-p) \end{aligned}$$

Finally, $T(V)(p) \leq T(V)(1-p)$, and the reverse inequality follows in the same way. □

Hence, without loss generality, we may assume that $0 \leq p \leq \frac{1}{2}$.
First remark that if $\mathbf{p} = \mathbf{0}$, we get obviously $p^+ = 0$ and $p^- = 0$ and so $T(V)(0) = V(\frac{1}{3}) = \beta_n$.
Now, we assume that $0 < p \leq \frac{1}{2}$, let us observe that $p \leq p^+ \leq 1$ and $0 \leq p^- \leq p$, hence equation $(1-p)p^+ + pp^- = p$ gives that $p^+ = \frac{p(1-p^-)}{(1-p)}$ and similarly $p^- = 1 - \frac{1-p}{p}p^+$. So, the set of $(p^+, p^-)$ verifying such constraints may be parametrized by the set of $p^+$ such that $p \leq p^+ \leq \frac{p}{1-p}$.
Since, $\frac{p^-}{3} + \frac{1}{3} = \frac{2}{3} - \frac{1-p}{3p}p^+$ and $p \neq 0$, $T(V)(p)$ becomes

$$T(V)(p) := \max_{p \leq p^+ \leq \frac{p}{1-p}} (1-p)p^+ + (1-p)V(\frac{p^+}{3}+\frac{1}{3}) + pV(\frac{2}{3} - \frac{1-p}{3p}p^+) \quad (6.3.3)$$

We remind that $V$ is piecewise linear, so optimal $p^+$ in (6.3.3) is such that $\frac{p^+}{3} + \frac{1}{3}$ or $\frac{2}{3} - \frac{1-p}{3p}p^+$ is equal to $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ or 1. Thus, we have just to compute all possibilities.
Furthermore, $\frac{p^+}{3} + \frac{1}{3}$ and $\frac{2}{3} - \frac{1-p}{3p}p^+$ are subject to the constraints

$$\frac{1}{3} \leq \frac{2}{3} - \frac{1-p}{3p}p^+ \leq \frac{p}{3} + \frac{1}{3} \leq \frac{p^+}{3} + \frac{1}{3} \leq \frac{1}{3(1-p)}$$

**Case 1 : $\mathbf{0 < p \leq \frac{1}{3}}$**
In this case, $\frac{1}{3} < \frac{p}{3} + \frac{1}{3} \leq \frac{4}{9} < \frac{1}{2}$ and $\frac{1}{3} < \frac{1}{3(1-p)} < \frac{1}{2}$ so no vertex is available for $\frac{p^+}{3} + \frac{1}{3}$. So $p^+ = \frac{p}{1-p}$ corresponds to the only vertex available for $\frac{2}{3} - \frac{1-p}{3p}p^+$, which is $\frac{1}{3}$. And so, for all $0 < p \leq \frac{1}{3}$

$$T(V)(p) = p + (1-p)V(\frac{1}{3(1-p)}) + pV(\frac{1}{3})$$

Let us observe first that on the interval $[\frac{1}{3}, \frac{1}{2}]$, $V$ is equal to

$$6(\gamma_n - \beta_n)(p - \frac{1}{3}) + \beta_n$$

Since $\frac{1}{3} < \frac{1}{3(1-p)} \leq \frac{1}{2}$, $V(\frac{1}{3(1-p)}) = 2(\gamma_n - \beta_n)\frac{p}{1-p} + \beta_n$, and so, we obtain

$$\begin{aligned} T(V)(p) &= p + (1-p)(2(\gamma_n - \beta_n)\tfrac{p}{1-p} + \beta_n) + p\beta_n \\ &= p + 2(\gamma_n - \beta_n)p + \beta_n \end{aligned}$$

In particular, $T(V)$ is linear on $[0, \frac{1}{3}]$ and

$$T(V)(0) = \beta_n, \text{ and } T(V)(\frac{1}{3}) = \frac{1}{3} + \frac{2}{3}\gamma_n + \frac{1}{3}\beta_n \qquad (6.3.4)$$

**Case 2 : $\frac{1}{3} < p < \frac{1}{2}$**

In this case, $\frac{4}{9} < \frac{p}{3} + \frac{1}{3} < \frac{1}{2}$ and $\frac{1}{2} < \frac{1}{3(1-p)} < \frac{2}{3}$ so we have to consider the two following sub-cases for $\frac{p^+}{3} + \frac{1}{3}$ :

$$\frac{p^+}{3} + \frac{1}{3} = \frac{1}{3(1-p)} \text{ or } \frac{p^+}{3} + \frac{1}{3} = \frac{1}{2}$$

Firstly, the case $\frac{p^+}{3} + \frac{1}{3} = \frac{1}{3(1-p)}$ corresponds to $\frac{2}{3} - \frac{1-p}{3p}p^+ = \frac{1}{3}$. And so, objective function evaluated at this point is equal to

$$a(p) := p + (1-p)V(\frac{1}{3(1-p)}) + pV(\frac{1}{3})$$

Let us observe first that on the interval $[\frac{1}{2}, \frac{2}{3}]$, $V$ is equal to

$$6(\gamma_n - \beta_n)(\frac{2}{3} - p) + \beta_n$$

Since $\frac{1}{2} < \frac{1}{3(1-p)} < \frac{2}{3}$, $V(\frac{1}{3(1-p)}) = 2(\gamma_n - \beta_n)\frac{1-2p}{1-p} + \beta_n$, and so, we obtain

$$\begin{aligned} a(p) &= p + (1-p)(2(\gamma_n - \beta_n)\tfrac{1-2p}{1-p} + \beta_n) + p\beta_n \\ &= p + 2(\gamma_n - \beta_n)(1-2p) + \beta_n \end{aligned}$$

Secondly, the case $\frac{p^+}{3} + \frac{1}{3} = \frac{1}{2}$ (i.e. $p^+ = \frac{1}{2}$) corresponds to $\frac{2}{3} - \frac{1-p}{3p}p^+ = \frac{5p-1}{6p}$. And so, objective function evaluated at this point is equal to

$$b(p) := \frac{(1-p)}{2} + (1-p)V(\frac{1}{2}) + pV(\frac{5p-1}{6p})$$

Let us remind that on the interval $[\frac{1}{3}, \frac{1}{2}]$, $V$ is equal to

$$6(\gamma_n - \beta_n)(p - \frac{1}{3}) + \beta_n$$

Since $\frac{1}{3} < \frac{5p-1}{6p} < \frac{1}{2}$, $V(\frac{5p-1}{6p}) = 2(\gamma_n - \beta_n)\frac{3p-1}{2p} + \beta_n$, and so, we obtain

$$\begin{array}{rcl} b(p) & = & \frac{(1-p)}{2} + (1-p)\gamma_n + p((\gamma_n - \beta_n)\frac{3p-1}{p} + \beta_n) \\ & = & \frac{(1-p)}{2} + 2p\gamma_n + \beta_n(1-2p) \end{array}$$

Finally, on the interval $[\frac{1}{3}, \frac{1}{2}]$, we get $T(V)(p) = \max(a(p), b(p))$.
Let us observe that, $a(\frac{1}{3}) = b(\frac{1}{3}) = \frac{1}{3} + \frac{2}{3}\gamma_n + \frac{1}{3}\beta_n$ which is exactly $T(V)(\frac{1}{3})$. Furthermore, $a$ and $b$ are linear, so $T(V)$ is linear on $[\frac{1}{3}, \frac{1}{2}]$ and

$$T(V)(\frac{1}{2}) = \max(a(\frac{1}{2}), b(\frac{1}{2})) = \max(\frac{1}{2} + \beta_n, \frac{1}{4} + \gamma_n) \tag{6.3.5}$$

**Case 3 : $\mathbf{p = \frac{1}{2}}$**
Since $T(V)$ is concave by theorem 6.2.3, this case follows immediately by the continuity in $\frac{1}{2}$ of $T(V)$.

Finally, $T(V)$ is linear on each intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{1}{2}]$, with

$$\left\{ \begin{array}{rcl} T(V)(0) & = & \beta_n \\ T(V)(\frac{1}{3}) & = & \frac{1}{3} + \frac{2}{3}\gamma_n + \frac{1}{3}\beta_n \\ T(V)(\frac{1}{2}) & = & \max(\frac{1}{2} + \beta_n, \frac{1}{4} + \gamma_n) \end{array} \right.$$

And so, since $V_0 = 0$, by theorem 6.2.2, we get recursively the value :

For al $n$ in $\mathbb{N}$, $V_n$ is piecewise linear on $[0,1]$ of vertices $(0, \alpha_n), (\frac{1}{3}, \beta_n), (\frac{1}{2}, \gamma_n), (\frac{2}{3}, \beta_n), (1, \alpha_n)$. Moreover, $\alpha_n$, $\beta_n$ and $\gamma_n$ verify the following recursive system

$$\left\{ \begin{array}{rcl} \alpha_{n+1} & = & \beta_n \\ \beta_{n+1} & = & \frac{1}{3}(1 + \beta_n + 2\gamma_n) \\ \gamma_{n+1} & = & \max(\frac{1}{2} + \beta_n, \frac{1}{4} + \gamma_n) \end{array} \right. \tag{6.3.6}$$

with $\alpha_0 = \beta_0 = \gamma_0 = 0$.
Then to prove the theorem 6.1.1, we have just to show that, for all $n$,

$$\frac{1}{2} + \beta_n \geq \frac{1}{4} + \gamma_n$$

Since it is obviously true for $n$ equal 0, and we have also that $\gamma_0 \geq \beta_0$. Hence, we prove recursively that :

$$\frac{1}{2} + \beta_n \geq \frac{1}{4} + \gamma_n \text{ and } \gamma_n \geq \beta_n$$

Which is equivalent to show inequalities

$$1/4 \geq \gamma_n - \beta_n \geq 0$$

If we assume that $1/4 \geq \gamma_n - \beta_n \geq 0$ then equation (6.3.6) gives

$$\gamma_{n+1} - \beta_{n+1} = 1/6 - 2/3(\gamma_n - \beta_n)$$

Finally, we obtain

$$1/4 > 1/6 \geq \gamma_{n+1} - \beta_{n+1} \geq 0.$$

And the theorem 6.1.1 follows.$\square$

Now, we focus our analysis on the value of infinitely repeated game. As reminded in introduction, we have just to analyze the asymptotic behavior of $\frac{V_n}{n}$. Hence, we have to provide the limit of $\frac{\alpha_n}{n}$, $\frac{\beta_n}{n}$,and $\frac{\gamma_n}{n}$. The following lemma gives the result.

**Lemma 6.3.1** *If $\alpha_n$, $\beta_n$ and $\gamma_n$ verify (6.1.1) for all $n$ then $\frac{\alpha_n}{n}$, $\frac{\beta_n}{n}$,and $\frac{\gamma_n}{n}$ converge to $\frac{2}{5}$ as $n$ goes to infinity.*

**Proof :** Let us denote $X_n := \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix}$, $D := \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}$ and $E := \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix}$, so we get

$$X_{n+1} = D + EX_n \tag{6.3.7}$$

Since $(3,2)E = (3,2)$, it is useful to put $y_n = 3\beta_n + 2\gamma_n$, then equation (6.3.7) gives

$$y_{n+1} = 2 + y_n$$

Finally, with $y_0 = 0$, we get $y_n = 2n$, and then $\frac{y_n}{n} = 2$.
Similarly, since $(-1,1)E = -\frac{2}{3}(-1,1)$, we put $z_n = \gamma_n - \beta_n$ and we find

$$z_{n+1} = \frac{1}{6} - \frac{2}{3}z_n$$

With $z_0 = 0$, we thus obtain

$$z_n = \frac{1}{6}\sum_{i=0}^{n-1}(-\frac{2}{3})^i = \frac{1}{10}(1 - (-\frac{2}{3})^n)$$

Let us observe that $z_n$ is bounded, so $\frac{z_n}{n}$ converges to 0 as $n$ goes to infinity. Finally, $\beta_n$ and $\gamma_n$ becomes

$$\beta_n = \frac{1}{5}(y_n - 2z_n) \text{ and } \gamma_n = \frac{1}{5}(y_n + 3z_n)$$

And thus, lemma follows :

$$\begin{cases} \frac{\beta_n}{n} \underset{n\to+\infty}{\longrightarrow} \frac{2}{5} \\ \frac{\gamma_n}{n} \underset{n\to+\infty}{\longrightarrow} \frac{2}{5} \end{cases}$$

And the convergence of $\frac{\alpha_n}{n}$ to $\frac{2}{5}$ follows immediately.$\square$

# Bibliographie

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# Perspectives

Cette étude est rédigée de façon à mettre en évidence trois types d'extensions du modèle de jeux financiers introduit par De Meyer et Moussa Saley : Le mécanisme d'échange, l'asymétrie et la diffusion de l'information. Ces trois axes d'études soulignent indépendamment différentes problématiques et soulèvent certaines questions ouvertes. Nous détaillons donc les perspectives envisagées distinctement pour chaque contexte d'études :

**Le mécanisme d'échange :**

Dans le contexte des jeux répétés à information incomplète d'un côté avec espaces d'actions finis (voir section 1.3), les études effectuées sur l'analyse du terme d'erreur mettent en évidence une classe de jeux pour lesquels la loi normale apparaît. Cependant, le chapitre 2 fournit un mécanisme d'échange particulier pour lequel le terme d'erreur est nul et souligne par la même, l'existence de différentes classes de jeux. De ce fait, déterminer une classification des mécanismes d'échange basée sur le terme d'erreur, serait une perspective envisageable.

**L'asymétrie d'information :**

L'étude du comportement asymptotique du jeu financier, dans le cadre d'une asymétrie bilatérale d'information, (chapitre 4) fait apparaître un jeu Brownien. Contrairement à l'approche de B. De Meyer dans "*From repeated games to Brownian games.*", le jeu limite est obtenu par convergence globale du jeu finiment répété. En effet, afin de préciser le terme d'erreur, l'utilisation d'EDP et les conditions de régularité ne sont pas nécessaires. Dans le but de compléter l'analyse du chapitre 4, une étude plus approfondie du jeu Brownien obtenu doit être effectuée. Cela nous amène à proposer différentes pistes de recherches à envisager :
Une première voie serait de déterminer la valeur et les stratégies optimales de ce jeu de façon directe. Afin d'analyser le comportement asymptotique du processus de prix dans le jeu initial, il suffirait, par l'intermédiaire des convergences effectuées dans le chapitre 4, de montrer que le processus limite est optimal dans le jeu Brownien.

Une seconde voie de recherche serait de déterminer la valeur de ce jeu Brownien par l'utilisation d'EDP. Cette étude fera apparaître des problématiques concernant la régularité de la valeur et le type de solutions à envisager. Ce qui pourrait permettre dans un dernier temps d'aborder numériquement le problème.

**La diffusion de l'information :**

Une perspective naturelle serait dans un premier temps d'approfondir l'étude des "Markov Chain Games" et par la suite, de l'appliquer au contexte financier.

# Appendice A : Jeux à somme nulle

Dans cet appendice, nous donnons les notions de base de la théorie des jeux à somme nulle.

## Définitions :

Un **jeu à somme nulle** est défini par un triplet $(g; X, Y)$. Les ensembles $X$ et $Y$ correspondent respectivement aux espaces de stratégies du joueur 1 et 2 et $g$ est une fonction de paiement définie sur le produit $X \times Y$. Dans le jeu que nous allons définir, le joueur 1 maximise et le joueur 2 minimise. Le choix $x \in X$ du joueur 1 et $y \in Y$ du joueur 2 détermine un paiement $g(x, y)$.
Nous dirons que le joueur 1 peut se **garantir** $\alpha$, si il existe une stratégie $x \in X$ telle que

$$\forall y \in Y, \quad g(x, y) \geq \alpha$$

En d'autres termes, si $\inf_{y \in Y} g(x, y) \geq \alpha$. Le joueur 1 peut donc se garantir au plus

$$\underline{v} = \sup_{x \in X} \inf_{y \in Y} g(x, y)$$

De même, le joueur 2 peut au plus se garantir la valeur :

$$\overline{v} = \inf_{y \in Y} \sup_{x \in X} g(x, y)$$

Nous remarquons que l'égalité suivante est toujours vérifiée : $\underline{v} \leq \overline{v}$.
Une stratégie du joueur 1 (resp. 2 ) garantissant $\underline{v}$ (resp. $\overline{v}$) est appelée **optimale**.
Un théorème de **minmax** est un théorème donnant des conditions garantissant l'égalité : $\overline{v} = \underline{v}$. Dans ce cas, nous dirons que le jeu $(g; X, Y)$ a une valeur $v = \overline{v} = \underline{v}$.

En supposant les bonnes propriétés de mesurabilité, nous introduisons $(\gamma, \Sigma, \mathcal{T})$ **l'extension mixte** du jeu $(g; X, Y)$, où $\Sigma := \Delta(X)$ et $\mathcal{T} := \Delta(Y)$ et $\gamma$ l'extension bilinéaire de $g$. Si $\sigma \in \Sigma$ et $\tau \in \mathcal{T}$,

$$\gamma(\sigma, \tau) = E_{\sigma \otimes \tau}[g]$$

# Appendice B : Théorème du Minmax

## Théorème de Sion et application :

Nous rappelons dans cette section l'énoncé des principaux résultats utilisés. Le théorème suivant est dû à Sion (1958).

**Theorem 6.3.2** *Soient $X$ et $Y$ des sous-ensembles convexes d'un espace vectoriel topologique, un des deux étant compact. Nous supposons que, pour tout $\alpha$ et pour tout couple $(x_0, y_0)$ dans $X \times Y$ les ensembles $\{x \in X; g(x, y_0) \geq \alpha\}$ et $\{y \in Y; g(x_0, y) \leq \alpha\}$ sont convexes et fermés.*
*Alors le jeu $(g; X, Y)$ a une valeur (et il existe une stratégie optimale pour le joueur ayant un espace de stratégie compact).*

Nous énonçons maintenant un théorème du minmax classique pour l'extension mixte d'un jeu.

**Theorem 6.3.3** *Soient $X$ et $Y$ des espaces compacts, $g$ une fonction bornée et mesurable sur $X \times Y$. Supposons de plus que pour tout $(x_0, y_0) \in X \times Y$, $g(x_0, .)$ est semi-continue inférieurement sur $Y$ et $g(., y_0)$ semi-continue supérieurement sur $X$. Alors :*

$$\sup_{\sigma \in \Delta_f(X)} \inf_{y \in Y} E_\sigma[g(., y)] = \inf_{\tau \in \Delta_f(Y)} \sup_{x \in X} E_\tau[g(x, .)]$$

*Le jeu sur $\Delta(X) \times \Delta(Y)$ a une valeur, les joueurs ont des stratégies optimales.*

# Appendice C : Dualité

## Dualité de Fenchel :

Nous supposons dans la suite que $f$ est une fonction définie sur $X = \mathbb{R}^n$ à valeurs dans $\mathbb{R} \cup \{-\infty\}$. Nous noterons également $Dom(f)$ le domaine de $f$ défini par l'ensemble $Dom(f) := \{x \in X | f(x) > -\infty\}$. La fonction sera toujours supposée **propre** : $Dom(f) \neq \emptyset$. La **conjuguée de Fenchel** de $f$ est définie par : pour tout $p \in \mathbb{R}^n$

$$f^*(p) = \inf_{x \in \mathbb{R}^n} \{\langle x, p\rangle - f(x)\}$$

Nous pouvons remarquer immédiatement que $f^*$ est concave est semi-continue supérieurement. Nous pouvons donc énoncer le théorème de Fenchel

**Theorem 6.3.4** *Si $f$ est concave et semi-continue supérieurement alors*

$$f = f^{**}$$

Nous définissons à présent la notion de sous-différentielle : La **sous-différentielle** de $f$ au point $x$ est définie par

$$\partial f(x) := \{p \in \mathbb{R}^n | f(y) \leq f(x) + \langle y - x, p\rangle, \forall y \in \mathbb{R}^n\}$$

Ce qui mène directement à la proposition suivante :

**Proposition 6.3.5**

$$p \in \partial f(x) \Leftrightarrow f(x) + f^*(p) = \langle x, p\rangle$$

## Remarque :

Si $f$ est concave sur un sous-ensemble convexe $C$ de $\mathbb{R}^n$, nous prolongeons naturellement la fonction $f$ à $\mathbb{R}^n$ en posant $f(x) = -\infty$ si $x \notin C$ et la conjugué de Fenchel de $f$ devient

$$f^*(x) = \inf_{x \in C} \{\langle x, p\rangle - f(x)\}$$

# Appendice D : Programmation linéaire

## Forme standard, primal, dual :

Soit $A$ une matrice $m$ lignes, $n$ colonnes ($m \leq n$), $b$ un $m$-vecteur colonne, $c$ un $n$-vecteur ligne, appelé le vecteur coût.
On écrit habituellement un programme linéaire sous la forme suivante :

$$(S) = \left\{ \begin{array}{c} min(cx) \\ Ax \leq b \\ x \geq 0 \end{array} \right.$$

Nous pouvons transformer les contraintes d'inégalités en contraintes d'égalités en rajoutant des variables muettes, dite "*variables d'écarts* " ; nous pouvons écrire l'inéquation $Ax \leq b$ sous la forme $Ax = b$.
Nous remarquons, premièrement, que dans le cas $m \leq n$, l'ensemble $D = \{x/Ax = b, x \geq 0\}$ est non vide. L'ensemble précédent a une position centrale dans l'étude des programmes linéaires. Nous pouvons définir les différentes notions de solution :

- Un point de $D$ est appelé "*solution réalisable*".
- Si $\hat{x} \in D$ , tel que :

$$\begin{array}{cc} max(cx) & = c\hat{x} \\ x \in D & \end{array}$$

On dit que $\hat{x}$ est une "*solution optimale*" de $(S)$.

Dans la suite de cette section, nous développons une méthode usuelle en programmation linéaire permettant de transformer le problème $(S)$, dit "primal", en un problème équivalent, dit "dual". Cette technique d'approche est particulièrement utile dans l'étude de la programmation linéaire paramétrique.

On appelle "*dual*" du programme linéaire $(S)$, le programme linéaire suivant :

$$(D) = \left\{ \begin{array}{c} max(yb) \\ yA \geq c \\ y \geq 0 \end{array} \right.$$

Dans cette écriture $y$ est le vecteur inconnu, est un $m$-vecteur ligne. La méthode du simplex permettra d'établir une équivalence entre le problème primal et le problème dual. Une notion prépondérante dans l'implementation d'un algorithme de résolution d'un programme linéaire est la notion de "*base*".

## Base, solution de Base des programmes linéaires :

Nous utilisons le problème initial sous la forme suivante

$$(S) = \begin{cases} min(cx) \\ Ax = b \\ x \geq 0 \end{cases}$$

Avec les contraintes sous forme d'égalités, tel que le système $Ax = b$ soit de rang maximal $m$, avec $m \leq n$.
On appelle "*base*" de ce programme linéaire un ensemble $J \subset \{1, \ldots, n\}$ d'indices de colonne tel que $A^J$ soit carrée et inversible.
En d'autres termes une base est un ensemble $J$ d'indices de colonnes de $A$ tel que l'ensemble des colonnes $A^j$ avec $j \in J$ constitue une base de l'espace vectoriel engendré par les colonnes de $A$ (dans $\mathbb{R}^m$).

**Definition 6.3.6** *A une base $J$ de $(S)$, nous associons la solution du système linéaire :*

$$\begin{cases} x^J & = & (A^J)^{-1}b & \\ x_j & = & 0 & j \notin J \end{cases}$$

*Avec $x^J := (x_j)_{j \in J}$. Cette solution est dite* "solution de base" *correspondant à $J$. Les $x_j$ avec $j \notin J$ sont appelées les variables* "hors base"

Naturellement, la connaissance d'une base particulière nous mène à reconsidérer l'écriture du programme linéaire. Nous introduisons par la suite la notion d'écriture canonique associée à une base.

On suppose que $A$ est de taille $(m, n)$ et de rang maximal $m$. Soit $J$ une base de $(S)$, nous souhaitons donner un problème équivalent à $(S)$, c'est à dire ayant la même valeur et les mêmes solutions optimales, étant considérer comme canonique pour la base $J$. En notant,

- $\hat{J} = \{1, \ldots, n\} \backslash J$
- $\hat{c} = c - c^J (A^J)^{-1} A$, appelé le "*coût réduit relatif à $J$* "

Avec la notation $x^J := (x_j)_{j \in J}$, nous avons la propriété suivante :

**Proposition 6.3.7** *$(S)$ équivalent a $(S^J)$.*
*Avec*

$$(S^J) = \begin{cases} min[\hat{c}^{\hat{J}} x^{\hat{J}} + c^J (A^J)^{-1} b] \\ x^J = (A^J)^{-1} b - (A^J)^{-1} A^{\hat{J}} x^{\hat{J}} \\ x \geq 0 \end{cases}$$

Nous faisons apparaître dans le nouveau programme linéaire la fonction objectif en fonction simplement des variables dites "hors base" et le système $(S^J)$ est appelé : la forme canonique de $(S)$ par rapport à la base $J$.

Dans cette partie nous donnons un complément du lexique employé en programmation linéaire. Premièrement, nous avons la notion de base réalisable :

**Definition 6.3.8** *Une base $J$ d'un programme linéaire $(S)$ est dite* "réalisable" *si la solution de base correspondante est réalisable. En d'autres termes si $(A^J)^{-1} b \geq 0$.*

La proposition suivante donne une idée plus fine de la méthode à employer pour résoudre de tels problèmes.

**Proposition 6.3.9** *Si le coût réduit $\hat{c}$ relatif à une base réalisable $J$ est positif ou nul, la solution de base correspondante est solution optimale du programme linéaire $(S)$.*

Nous savons donc, qu'il est suffisant de trouver une base ayant des coûts réduits positifs ou nuls pour déterminer une solution de notre problème. Nous donnons à de telles bases la terminologie suivante :

**Definition 6.3.10** *Une base $J$, tel que le vecteur coût relatif à $J$ est posittif ou nul, est dite* "base optimale"

La méthode du simplex est la procédure classique pour résoudre ce type de problèmes. Nous ne décrivons pas les détails de cette approche, mais nous citons un des principaux résultats qu'elle induits. Nous considérons toujours le problème initial $(S)$ et le problème dual $(D)$.

**Theorem 6.3.11** *Si deux programmes linéaires duaux $(S)$ et (D) ont l'un et l'autre une solution réalisable, ils ont l'un et l'autre une solution optimale et les valeurs des fonctions objectifs à l'optimum sont égales.*

# Notations

Pour $x$ dans $\mathbb{R}^n$ nous notons
$\|\|_{\mathbf{1}}$ : la norme 1 définie par $\|x\|_1 = \Sigma_i |x_i|$.
$\|\|_\infty$ : la norme $\infty$ définie par $\|x\|_\infty = \sup_i |x_i|$.

$\mathbf{\Delta(K)}$ : pour un ensemble $K$ fini, correspond au simplex des probabilités sur $K$. Plus généralement, $\Delta(X)$ pour un espace topologique $X$, est l'ensemble des probabilités régulières munis de la topologie faible*.

$\mathbf{\Delta_f(K)}$ : sous-ensemble de $\Delta(K)$ des probabilités à support fini.

**conv** : L'enveloppe convexe.

$\overline{\mathbf{A}}$ : La fermeture de l'ensemble $A$.

**intA** : L'intérieur de l'ensemble $A$.

$\mathbb{1}_{\mathbf{A}}$ : L'indicatrice de l'ensemble $A$.

**cav(f)** : le concavifié de la fonction $f$.

**vex(f)** : le convexifié de la fonction $f$.

$\partial\mathbf{f}$ : la sous-différentielle de $f$.

$\nabla\mathbf{f}$ : le gradient de la fonction $f$.

$\langle , \rangle$ : Le produit scalaire.

$\mathbf{E}_\pi[.]$ : L'espérance sous la probabilité $\pi$.

$\square$ : Fin de preuve.

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**Résumé :** Les problèmes de gestion optimale de l'information sont omniprésents sur les marchés financiers (délit d'initié, problèmes de défaut, etc). Leurs études nécessitent une conception stratégique des interactions entre agents : les ordres placés par un agent informé influencent les cours futurs des actifs par l'information qu'ils véhiculent. Cette possibilité d'influencer les cours n'est pas envisagée par la théorie classique de la finance. Le cadre naturel de l'étude des interactions stratégiques est la théorie des jeux. Cette thèse a précisément pour objet de développer une théorie financière basée sur la théorie des jeux. Nous prendrons comme base l'article de De Meyer et Moussa Saley , *"On the origin of Brownian Motion in finance"*. Cet article modélise les interactions entre deux teneurs de marché asymétriquement informés sur le futur d'un actif risqué par un jeu répété à somme nulle à information incomplète. Cette étude montre en particulier que le mouvement Brownien, souvent utilisé en finance pour décrire la dynamique des prix, a une origine partiellement stratégique : il est introduit par les acteurs informés afin de tirer un bénéfice maximal de leur information privée. Cette thèse traite de diverses extensions de ce modèle concernant l'influence de la grille des prix, l'asymétrie bilatérale d'information, le processus de diffusion de l'information.

**Mots-clés :** Jeux non-coopératifs, jeux répétés, information incomplète, jeu dual, terme d'erreur, jeu Brownien, programme linéaire paramétrique.

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**Abstract :** The problem of the optimal use of private information is omnipresent on the financial markets (Insider trading, problems of default, etc). To analyze such a problem properly, the interactions between agents are to be considered strategically : the information conveyed by the prices fixed by an informed agent influences the future behavior of the asset price. This opportunity of influencing price is generally not considered by the classical finance theory. Game theory is the natural framework to analyze strategically these interactions. The main aim of this thesis is precisely to develop financial theory based on game theory. De Meyer and Moussa Saley, in *"On the origin of Brownian Motion in finance"*, model the interactions between two asymmetrically informed market makers by a zero-sum repeated game with lack of information on one side. In particular, this study shows that the Brownian motion, often used in finances to describe the price dynamics, has partially a strategic origin : it is introduced by the informed agents in order to take maximal benefit from their private information. This thesis deals with several generalizations of this model about : Influence of the price grid, the bilateral asymmetry of information and the diffusion process of the information.

**Keywords :** Insider trading, noncooperative game, repeated games, incomplete information, dual game, error term, Brownian games, parametric linear programming.